Exercise 112
Determination of Earth's gravity using Kater's pendulum

March 12, 2009

The aim of this experiment is to determine the Earth gravitational acceleration using Kater's pendulum.

1 Theoretical perquisites

1. Gravity. Gravitational acceleration.
2. Rigid body dynamics.
3. Harmonic motion.
4. Physical pendulum. Derivation of oscillation period of physical pendulum for small oscillations.

2 Equipment

1. Kater's pendulum.
2. Stopwatch.
3. Calliper for measuring the distance between the suspension edges.

3 Measurement method

One method of determining gravitational acceleration is the use of Kater's pendulum[1], which is a special case of a physical pendulum. In case of small oscillation, the motion of such a pendulum can be described by the following equation

$$\frac{d^2\varphi}{dt^2} = -\frac{mgd}{I}\varphi = -\omega^2\varphi. \tag{1}$$

where m is the pendulum mass, d is the distance between the suspension point and the centre of mass S, I is the rigid-body moment of inertia, measured with respect to the suspension point. The period of the oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}. \tag{2}$$

[1]It was designed and built by British physicist Captain Henry Kater in 1817. For about a century it remained the standard method for measuring local gravitational acceleration during geographical surveys.

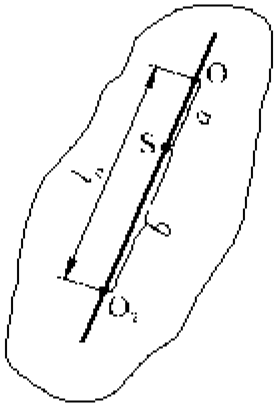

Figure 1: Physical pendulum with two suspension points.

Gravitational acceleration can be determined directly from the above relation. However, this would not be sufficiently exact (*why?*). Much better approach is determination of pendulum's *reduced length*. The pendulum's reduced length l_r is the length of a mathematical pendulum whose period of oscillations is equal to the period of the given physical pendulum, i. e.

$$T = 2\pi\sqrt{\frac{I}{mgd}} = 2\pi\sqrt{\frac{l_r}{g}}, \tag{3}$$

what is a direct consequence of Eq. 2 and well known expression for the period of mathematical pendulum. The above gives

$$l_r = \frac{I}{md}. \tag{4}$$

It can be shown that each physical pendulum has two suspension points giving the same period of oscillations and the distance between them is equal to the reduced length of the pendulum. Consider a physical pendulum (Fig. 1) suspended in point O_1, Its period is equal to

$$T_1 = 2\pi\sqrt{\frac{I_1}{mga}}. \tag{5}$$

Using Steiner's theorem the above equation can be written as

$$T_1 = 2\pi\sqrt{\frac{I_S + ma^2}{mga}}, \tag{6}$$

where I_S is the moment of inertia with respect to the axis parallel to the axis at O_1 and going through the centre of the mass S. Let the axis O_2 be parallel to O_1 and located at the line O_1S. The period of oscillations on this axis is

$$T_2 = 2\pi\sqrt{\frac{I_S + mb^2}{mgb}}, \tag{7}$$

where a and b are depicted in Fig. 1. Assume that periods T_1 and T_2 are equal. This is possible only if

$$\frac{I_S + ma^2}{mga} = \frac{I_S + mb^2}{mgb}, \tag{8}$$

which—after some algebra—gives

$$(b - a)\,(I_S - mab) = 0. \tag{9}$$

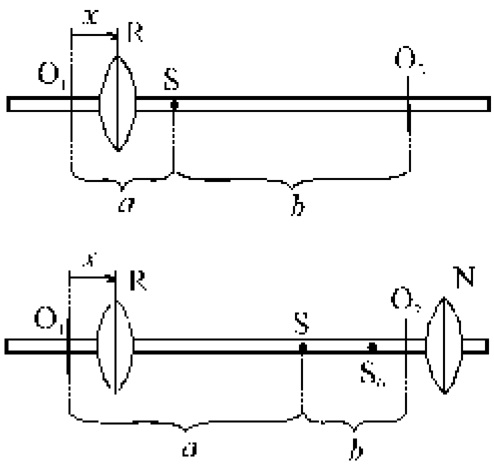

Figure 2: Symmetrical (top) and asymmetrical (bottom) reversible pendulum.

Eq. 9 means that the condition $T_1 = T_2$ can be fulfilled in one of the two cases: (I) when $a = b$, (II) if axis O_2 is chosen in such a point that

$$I_S = mab. \tag{10}$$

Situation I corresponds to the equal periods of identical pendulum, which is a trivial and non-interesting case. For case II, Eqs. (6), (7), and (10) give

$$T_1 = T_2 = 2\pi\sqrt{\frac{mab + mb^2}{mgb}} = 2\pi\sqrt{\frac{a+b}{g}}. \tag{11}$$

This compared with Eq. (3) shows that $l_r = a + b$. Then

$$T_1 = T_2 = 2\pi\sqrt{\frac{l_r}{g}}. \tag{12}$$

We have shown that the distance between suspension axes—equal to $a + b$—is the reduced length of the physical pendulum. This suggest the precise method of determination gravitational acceleration, without the need to estimate I_s, m, nor location of the centre of the mass. As finding the location of the axes O_1 and O_2 can be a challenge, the pendulum has additional weight attached, which can be used to alter location of the centre of the its mass S and its moment of inertia I_s. It is use to adjust these properties to fulfil the condition $T_1 = T_2$.

Fig. 2 shows a construction of such a pendulum. It consists of a metal rod with two knife edges for suspending and two weights, one of which is movable. By measuring oscillation periods of the pendulum suspended at both knife edges one can adjust the position of the weight R to fulfil the condition $T_1 = T_2$. However, in case of symmetrical pendulum, it is very hard to estimate whether the equality of periods means the trivial situation I (i. e. $a = b$) or the desired situation II. The solution to this issue is use of asymmetric pendulum which has its centre of mass very close to one of the axes (e. g. O_2). In such a case the undesired case of $a = b$ can be avoided.

By adjusting the position of movable mass R it is possible to find the equal periods, corresponding to Eq. (10). As $l_r = a + b$, Eq. (10) can be expressed as

$$I_S = ma\left(l_r - a\right), \tag{13}$$

where a is an adjusted parameter, which changes the value of I_S. Because Eq. (13) is a quadratic equation, one can expect to have two positions for which $T_1 = T_2$. In such a case gravitational acceleration can be estimated as

$$g = \frac{4\pi^2 l_r}{T^2}, \tag{14}$$

where $T = T_1 = T_2$ and l_r is the distance between the knife edges.

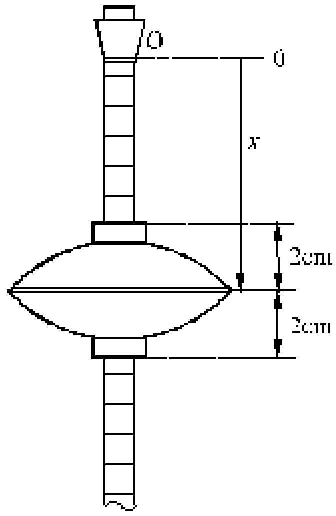

Figure 3: Illustration of measurements of distance x between movable mass R and axisO_1.

4 Measurements

In order to determine gravitational acceleration using Kater's pendulum, you have to adjust the position of mass R. You begin near the axis O_1 and measure oscillation periods for pendulum suspended on both axes. Next you shift the mass by 1 cm or 2 cm (according to the markings) and repeat the measurements. You have to note the position x of the moving mass as the distance between the centre of the mass R and the mark at knife edge O_1 (see Fig. 3). The thickness of the mass is 2 cm and of the nut 1 cm.

In the experiment you have to note periods T_1 and T_2 for each of the marked positions x of the movable weight. Considering the fact, that a precision of time measurements with a stopwatch is no better than 0.2 s–0.5 s, you should find the way to measure the oscillation period with precision of at least 0.02 s. The mass R can be shifted only after releasing the nut. During the measurements the nut must be locked. Take care that oscillations amplitude is not too high (max. 5°), otherwise the oscillations are not harmonic. On the other hand, the oscillations cannot be too small due to the—very small although present—damping and friction.

Your task is to estimate such position of mass R, for which both periods are equal. They can be found by making plots of the measured periods T as a function of the movable mass position x for both axes (see Fig. 4). The desired position is the one, at which the functions $T_1(x)$ and $T_2(x)$ coincide. Usually it lies somewhere between the marked measurement points. In such a case the proper period $T = T_1 = T_2$ can be estimated with one of the two methods:

1. The experimental points closest to the cross-sections can be linearly interpolated and the desired T computed as vertical co-ordinate of the crossing of two straight lines.

2. All the experimental points for each suspension points separately can be approximated by third- of fourth-order polynomial with least squares method. Next the crossing points can be found numerically as roots of the function $f(x) = P_1(x) - P_2(x)$, where $P_1(x)$ and $P_2(x)$ are approximated polynomials.

As already stated, there are two points A and B at which $T_1 = T_2$. In consequence the periods in these points, T_A and T_B, respectively should be equal to each other (*which one do you think is more precise?*) and correspond to the Earth gravitational acceleration g through Eq. 14. This equation allows to compute g with an absolute error

$$\Delta g = g\left(\frac{\Delta l_r}{l_r} + \frac{2\Delta T}{T}\right), \tag{15}$$

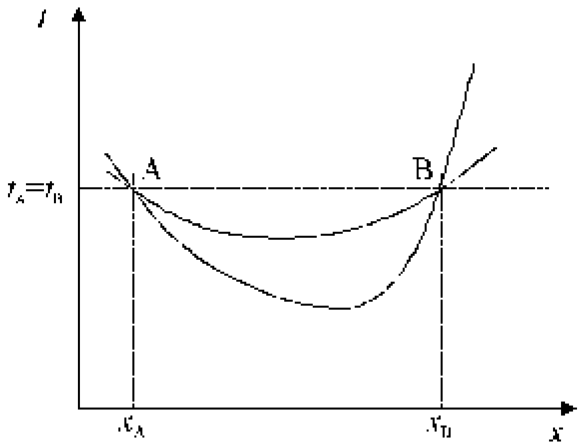

Figure 4: Plot of relation $T(x)$ for the pendulum suspended from both axes. Horizontal coordinates of the cross-section points fulfil the conditions: $T_A = T_1(x_A) = T_2(x_A)$, $T_B = T_1(x_B) = T_2(x_B)$.

where Δl_r and ΔT are errors of estimation of l_r and T, respectively.

During the analysis of your results and preparation of the report pay a special attention to the possible sources of errors. What effects can play some role and have not been considered? Compare the determined value of g with tabular data and conclude on the accuracy and precision of the experimental method.