Experiment 123
Determination of the sound wave velocity with the method of Lissajous figures

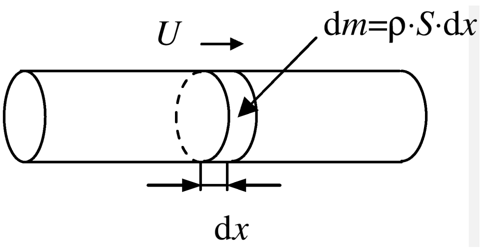
The aim of the exercise
1. To study acoustic wave propagation in the air
2. To determine of the sound wave velocity in the air

Methodology of the measurement
The velocity of the longitudinal wave in the solid states can be obtained as follows:
Assume that the rod of the density ρ is extended along the *x*-axis. Consider a small cylinder of a length d*x* and a cross-section area *S*. The relative extension of the bar equals:

$\left(\frac{\partial U}{\partial x}\right)_x$ and

$$\left(\frac{\partial U}{\partial x}\right)_{(x+dx)} = \left(\frac{\partial U}{\partial x}\right)_x + \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)_x \cdot dx = \left(\frac{\partial U}{\partial x}\right)_x + \left(\frac{\partial^2 U}{\partial x^2}\right)_x \cdot dx.$$

in position *x* and *x*+d*x*, respectively.



For systems that obey Hooke's law, the relative extension equals:

$$\frac{\partial U}{\partial x} = \frac{1}{E}\cdot\frac{F}{S},$$

where *E* is the modulus of elasticity and *F* the restoring force exerted by the material. The resultant force acting on the mass d*m* of the cylinder reads:

$$F_x = E\cdot S\cdot\left[\left(\frac{\partial U}{\partial x}\right)_{x+dx} - \left(\frac{\partial U}{\partial x}\right)_x\right] = E\cdot S\cdot\left(\frac{\partial^2 U}{\partial x^2}\right)\cdot dx.$$

According to the second Newton's law the resultant force is given by the expression:

$$F_x = dm\cdot\frac{\partial^2 U}{\partial t^2} = \rho\cdot S\cdot dx\cdot\frac{\partial^2 U}{\partial t^2}.$$

Comparing both equations one can derive the following differential equation:

$$\frac{\partial^2 U}{\partial t^2} = \frac{E}{\rho}\cdot\frac{\partial^2 U}{\partial x^2}.$$

which is a classical propagating wave equation:

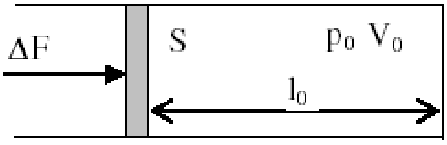
$$\frac{\partial^2 U}{\partial t^2} = \left(c^2\right)\frac{\partial^2 U}{\partial x^2},$$

where *c* refers to the propagation speed of the longitudinal perturbation and equals:

$$c = \sqrt{\frac{E}{\rho}}.$$

For the acoustic wave propagating in the air one has to find the equivalent of the elasticity modulus for a gas. Hence we can consider a cylinder with a gas and a moving piston with a cross-section area S. The starting pressure of the gas is given by p_0 and the piston exerts force

ΔF on the gas, which produces additional pressure $\Delta p = \Delta F/S$. As a result the starting volume of the gas will be reduced. The initial and final lengths of the cylinder fulfilled with the gas equal: $l_0 = V_0/S$ and $l_0 - \Delta l = V_0/S - \Delta V/S$, respectively.



The Hooke's law for the gas reads:

$$\Delta p = \frac{\Delta F}{S} = -E \cdot \frac{\Delta l}{l_0} = -E \cdot \frac{\Delta l \cdot S}{l_0 \cdot S} = -E \cdot \frac{\Delta V}{V_0}.$$

where $E = -\frac{\Delta p \cdot V_0}{\Delta V} \cong -V_0 \cdot \frac{dp}{dV} = K$, and K is the modulus of gas compressibility.

Under the assumption that fluctuations of the gas are fast enough to disable the heat transfer the gas undergoes the adiabatic process: $p \cdot V^{\kappa} = \text{const}$, where $\kappa = c_p/c_V$, c_p being the specific heat for constant pressure and c_V being the specific heat for constant volume. Differentiating the above-mentioned expression $\left(V^{\kappa} \cdot dp + \kappa \cdot p \cdot V^{\kappa-1} \cdot dV = 0\right)$ and dividing the result by $V^{\kappa-1}$ yield: $\frac{dp}{dV} = -\kappa \cdot \frac{p}{V}$,

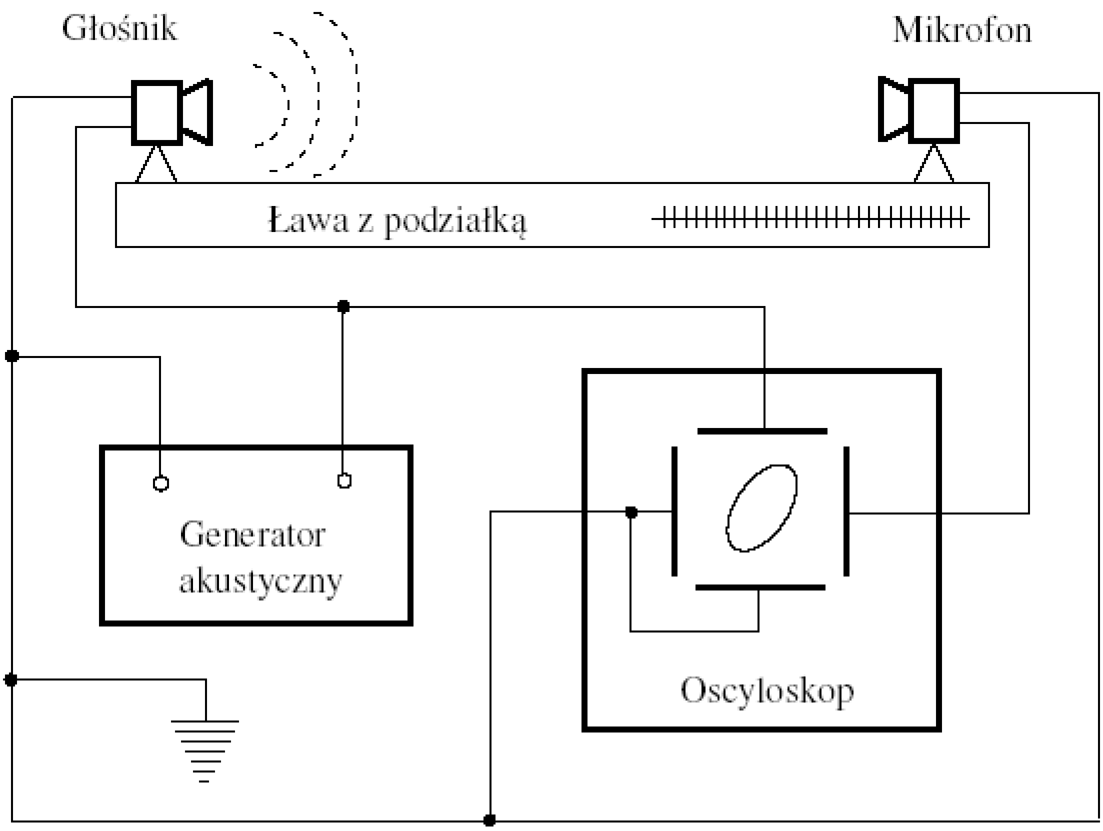
Combining this equation with the speed of sound equation one obtains the formula

$$c = \sqrt{\frac{\kappa \cdot p}{\rho}}.$$

that describes the speed of sound in a gas.

The main components of the air are diatomic gases (N_2, O_2) hence $\kappa = 1.4$. Typical atmospheric pressure equals 10^5 N/m^2. For such conditions the density of air equals 1.3 kg/m^3 and, consequently, the speed of sound in the air equals 328 m/s.

The measurement process



The measurement of the speed of sound in the air can be performed in the experimental setup presented above. The signal emitted by the acoustic generator is transmitted to the speaker and to the horizontal plates of the oscilloscope. The acoustic wave propagates toward the microphone and is transformed into the electric signal transmitted to the vertical plates of the oscilloscope. The microphone can be moved on the bench and the position of the microphone can be read from the tape-measure. The signal from microphone is shifted with respect to the signal from the speaker according to the relation:

$$\Phi = \Phi_0 + \Phi(r),$$

where Φ_0 stands for the phase shift related to all phenomena not related to the sound waves, $\Phi(r)$ is a phase shift determined by the positions of the speaker and the microphone.
If one applies the harmonic signal to the horizontal and vertical plates of the oscilloscope, namely $y(t) = B \cdot \sin(\omega \cdot t)$, and $x(t) = A \cdot \sin(\omega \cdot t + \Phi)$, the resultant figure observed on the screen will be described by the following expression

$$\frac{x^2}{A^2} - 2 \cdot \frac{x \cdot y}{A \cdot B} \cdot \cos(\Phi) + \frac{y^2}{B^2} = \sin^2(\Phi).$$

The above equation corresponds to the ellipse. For the phase shift $\Phi = k\pi$, where k is the integer, the equation below describes the line

$$\left(\frac{y}{B} \pm \frac{x}{A}\right)^2 = 0 .$$

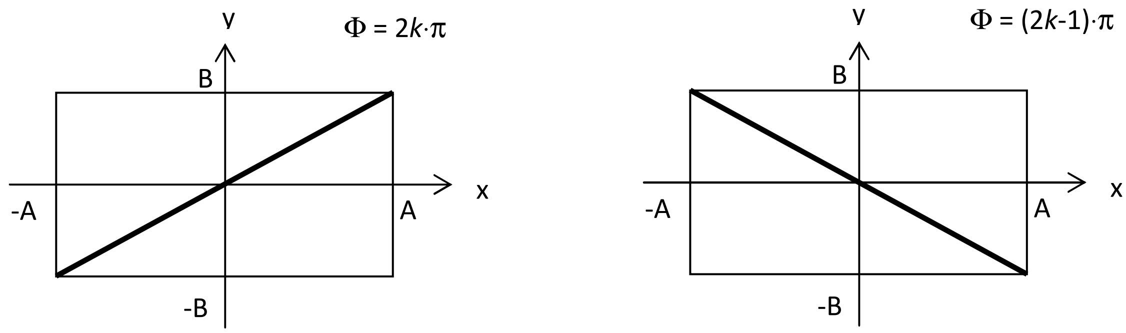
Therefore, for $\Phi = 2k\pi$ one obtains

$$y = \frac{B}{A} \cdot x \quad ,$$

and for $\Phi = (2k - 1)\,\pi$

$$y = (-1) \cdot \frac{B}{A} \cdot x \quad .$$

The illustrative example is given below



The transition from one figure to another can be realized by moving the microphone with respect to the speaker, which results in the phase shift $\Phi = \pi$. The minimal distance between the positions corresponding to both figures is D, which is related to the wave period T, the wavelength λ and the frequency f by means of the formula:

$$D = \frac{\lambda}{2} = \frac{c \cdot T}{2} = \frac{c}{2 \cdot f} ,$$

Performing a number of the measurements of D for a given frequency f one can determine the velocity of sound in the air:

$$\boxed{c = 2 \cdot f \cdot D}$$

Procedure

1. Turn on the oscilloscope and the generator.
2. Set the sinusoidal signal with the maximal frequency of $f = 2.5$ kHz on the generator.
3. Measure the frequency of the signal using the oscilloscope. Estimate the error Δf. Introduce the results to the table.
4. Move the microphone to the extreme far position and observe the figures on the screen. There should be an ellipse inscribed within 2 cm x 1 cm rectangular on the screen.
5. Move microphone back till the line appears on the screen. Introduce the position of the microphone into the table.
6. Move the microphone further and record the positions of the microphone corresponding to the line appearing on the screen.
7. Reverse the direction of moving the microphone and record the characteristic positions.

8. Change the frequency of the acoustic wave following the advice of the instructor and repeat points 3 – 7
9. Enter all the data in the table.
10. Estimate the error of the position of the microphone Δm.

Report preparation

The report should contain:
1. A short description of the experimental method.
2. The experimental data collected in the table.

Signal from generator		Number of the measurement	Position of the microphone	Distance	$c + \Delta c$
f [Hz]	Δf [Hz]	I	r_i [m]	$D = r_{i+1} - r_i$ [m]	[m/s]
		1			
		2			
		3			
		...			
		1			
		2			
		3			
		...			

3. Calculations and the error analysis
   For each set of D measurements and for each frequency f calculate:
   a) Arithmetic mean of D.
   b) Mean squared error according to the formula:

$$\Delta D = \sqrt{\frac{\sum_{i=1}^{n} [D - D_i]^2}{n \cdot (n-1)}},$$

   c) The measurement error
   $\Delta D_\alpha = t_\alpha \Delta D$
   using Student – Fisher error with α = 0.95
   d) The speed of the acoustic wave in the air according to the expression:
   $\bar{c} = 2 f \bar{D}$
   e) The error of the sound speed according to the formula:
   $\Delta c = 4\left(\bar{D}\Delta f + f\,\Delta D_\alpha\right)$
   f) The final speed for each frequency should be given in the form:
   $c = \bar{c} \pm \Delta c$
4. Conclusions