Analytical mechanics

Notes for students of Science and Technology

by Grzegorz W. Bąk

Lodz University of Technology, 2013

PREFACE

The basic course of experimental physics is usually subdivided into mechanics, thermodynamics and molecular physics, electricity and magnetism, optics and atomic and nuclear physics. Mechanics is the initial part of course of general physics, because the other parts cannot be studied without description of motion and its causes. Mechanics contains ***kinematics*** which describes motion of bodies considered irrelative to the factors causing the motion, ***dynamics*** which studies the laws of motion and the causes producing the motion and changing it and ***statics*** which deal with the state of equilibrium of the bodies.

The dynamics of a single point-like particle is given in Newtonian mechanics by the vector equation:

$$\frac{d\vec{p}}{dt} = \vec{F}$$

where $\vec{p} = m\vec{v}$ is the momentum and $\vec{F}$ is the force. The above equation results from experiment and cannot be derived from other equations or laws. The above vector equation can be written in the form of three scalar equations:

$$\frac{dp_x}{dt} = F_x$$

$$\frac{dp_y}{dt} = F_y$$

$$\frac{dp_z}{dt} = F_z$$

In order to solve a mechanical problem and find the motion resulting from the force $\vec{F}$ acting on a body we have to solve the above equations which can be sometimes a difficult task, especially when we have to do with a constrained motion. Analytical mechanics offers handy methods to solve many such mechanical problems. This introductory course encompasses an elementary understanding of analytical mechanics, especially the Lagrangian formulation of dynamics of motion. The Hamiltonian formulation (Chapter 3) is necessary for the connection between the Newtonian mechanics and quantum mechanics.

It will be assumed that students know and understand the basic concepts and mathematical methods within the scope of the first year course of basic physics, calculus and algebra. In the first few lectures some basic mechanical concepts will be recalled.

Basic bibliography:

W.Rubinowicz, W.Królikowski, *Mechanika Teoretyczna*, PWN Warszawa

G.W.Bąk, *Analytical Mechanics, Notes for students of Science and Technology*

CONTENT

1. BASIC CONCEPTS AND IDEAS

Length and time – Length measures the extension of bodies and time is a measure of duration of processes and phenomena. The definition of these quantities is a philosophical task to some extend and we shall assume in these lectures that the two physical quantities are clear and well understood.

A point-like particle – a point of negligible size but possessing mass. The concept of point-like particle is usually an approximation. Such an approximation can be used to describe the motion of the Earth around the Sun, but proves to be useless when we are to describe the motion of a table-tennis ball.

A position of a point-like particle can be described in relation to **a frame of reference** by its **radius-vector**, as shown in Fig.1.1. Position of a point-like particle is given by a vector

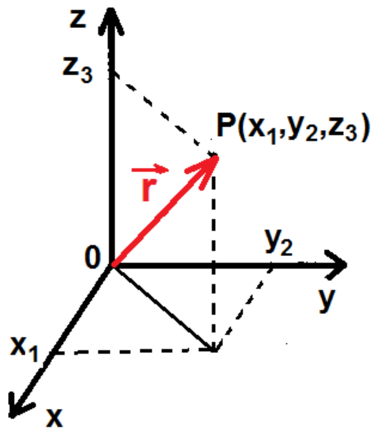

Fig.1.1. Position of a point-like particle is described in relation to a frame of reference by a vector $\vec{r} = \vec{i}x_1 + \vec{j}y_1 + \vec{k}z_1$*. Selection of an appropriate frame of reference may play a significant (vital) role to find a comparatively easy way to solve a mechanical problem.*

$$\vec{r} = \vec{i}x_1 + \vec{j}y_1 + \vec{k}z_1 \qquad (1.1)$$

in relation to a frame of reference consisting of three mutually perpendicular coordinate axes.

Let us note that the above equation is written under the assumption that:

- Physical space is three-dimensional. This assumption works well in classical physics, but is not valid in the theory of relativity.
- It is possible to define the position of a point-like particle accurately. This assumption is not valid in microphysics using quantum description of a micro-particles. According to the (Heisenberg) uncertainty principle it is not possible to find accurately both the position and the momentum of a particle.

As it results from the above short discussion, the assumption about 3D physical space has a deep physical meaning.

Movement and trajectory are described by the time-dependence of the position vector:

$$\vec{r} = \vec{r}(t) \qquad (1.2)$$

The radius-vector in Cartesian coordinates can be written in the form:

$$\vec{r}(t) = \vec{i}x(t) + \vec{j}y(t) + \vec{k}z(t) \qquad (1.3)$$

The equation (1.3) enables to write the parametric equations of trajectory:

$$x = x(t)$$
$$y = y(t) \qquad (1.4)$$
$$z = z(t)$$

We assume that the functions (1.4) are differentiable twice.

Velocity of a point-like particle is given by:

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} \qquad (1.5)$$

So we have:

$$\vec{v} = \frac{d}{dt}\left(x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}\right) = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} \qquad (1.6)$$

The distance covered by a particle is equal to the length of a trajectory curve and is given by:

$$s = s(t) = \int_{t_0}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \qquad (1.7)$$

Let us consider the expression $d\vec{r}/ds$ (see Fig.1.2). $d\vec{r}$ is a vector tangent to trajectory if $|d\vec{r}|$ approaches zero. The quantity $d\vec{r}/ds$ is therefore a unit vector tangent to trajectory of motion.

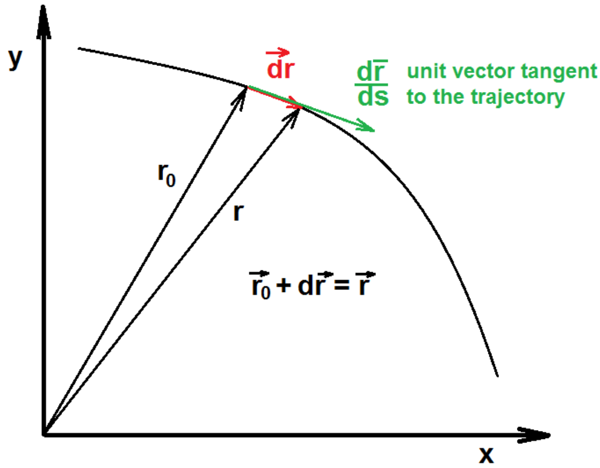

Fig.1.2. The vector $\frac{d\vec{r}}{ds}$ *is tangent to the trajectory and this is a unit vector because the length of the differential change of the position vector* $d\vec{r}$ *for the differential change of time dt is equal to the differential length of the distance covered by a moving point.*

The radius vector can be regarded as a composition of functions, so we can write $\vec{r} = \vec{r}(s(t))$.

Using the tangent vector $\frac{d\vec{r}}{ds}$ defined in Fig.1.2 we can write for the velocity:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \vec{t}\,v \qquad (1.8)$$

$\vec{t}$ is the unit vector tangent to the trajectory at the point of movement and v is the speed of a moving particle, s is the length of curve covered by a particle. It results from (1.8) that velocity is tangent to trajectory for any curvilinear motion.

Acceleration of a particle is defined as:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}\right) = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k} \tag{1.9}$$

1.1. Tangential and normal acceleration

Let us assume we have to do with a curvilinear motion (see Fig.1.3).

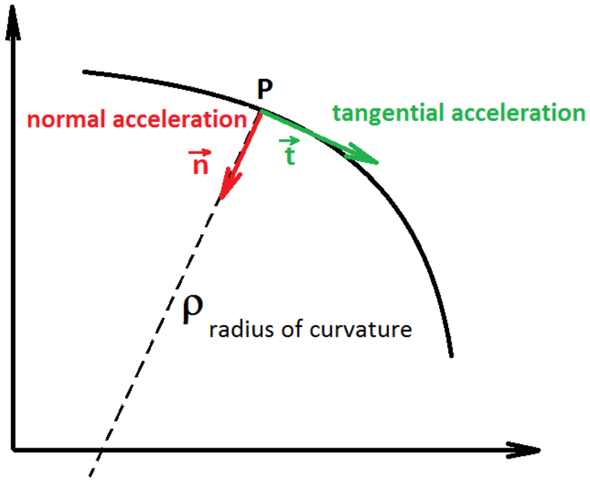

Fig.1.3. Trajectory of a curvilinear motion. ρ is the radius of curvature of the trajectory of motion at the point P. $\vec{n}$ and $\vec{t}$ are unit vectors normal and tangent to the curve at the point P.

The so-called Frenet formula:

$$\frac{d\vec{t}}{ds} = \frac{\vec{n}}{\rho} \tag{1.10}$$

where ds is the differential length of trajectory covered. $d\vec{t}/ds$ describes the change of direction of the unit vector $\vec{t}$ which is inversely proportional to the radius of curvature ρ.

Let us calculate the acceleration of motion for the curvilinear motion depicted in Fig. 1.3.

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\vec{t}\right) = \frac{d}{dt}\left(v\vec{t}\right) = \dot{v}\vec{t} + v\frac{d\vec{t}}{dt} = \\ &= \dot{v}\vec{t} + v\frac{d\vec{t}}{ds}\frac{ds}{dt} = \dot{v}\vec{t} + \frac{v^2}{\rho}\vec{n} \end{aligned} \tag{1.11}$$

As shown above the tangential component of acceleration is equal to:

$$a_t = \dot{v} \tag{1.12}$$

and the normal component of acceleration equals:

$$a_n = \frac{v^2}{\rho} \tag{1.13}$$

In case of a uniform motion of a point in a circle the normal acceleration is given by the well known formula:

$$a_n = \frac{v^2}{R} \tag{1.14}$$

where R is the radius of the circle.

1.2. Radial and transversal velocity and acceleration

Let us suppose that we have to do with a plane motion described in a fixed frame of reference (see Fig. 1.4). Let us assume that at a certain moment the position of a moving point is given by a radius-vector $\vec{r}$. We want to find the components of the acceleration of the point along the direction parallel to the radius-vector $\vec{r}$ (the radial component) and parallel to the direction perpendicular to the direction of the radius vector $\vec{r}$ (the transversal component). In order to solve the problem let us use the concept of complex plane. The position of a point on a plane can be written as:

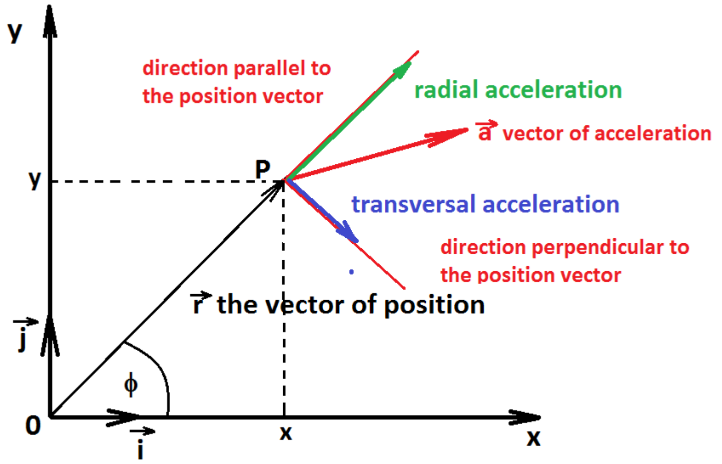

Fig.1.4. Radial and transversal components of acceleration of a particle at the point P.

$$\vec{r} = x\vec{i} + y\vec{j} \tag{1.15}$$

Using the complex plane we can write the position of the point P as:

$$z = x + iy = re^{i\varphi} \tag{1.16}$$

where $i = \sqrt{-1}$ is the imaginary unit and ϕ is the angle shown in Fig.1.4. Differentiating (1.16) with respect to time we obtain:

$$\begin{aligned} \dot{z} &= \dot{r}e^{i\varphi} + ir\dot{\varphi}e^{i\varphi} = (\dot{r} + ir\dot{\varphi})e^{i\varphi} \\ \ddot{z} &= \left\{\ddot{r} - r\dot{\varphi}^2 + i(2\dot{r}\dot{\varphi} + r\ddot{\varphi})\right\}e^{i\varphi} \end{aligned} \tag{1.17}$$

Taking the real and imaginary parts of the above equations we would get the x and y components of acceleration respectively. In order to obtain the radial and transversal components of acceleration we must rotate the frame of reference by the angle ϕ as shown in

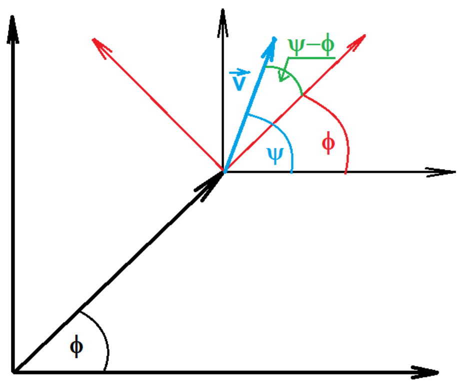

Fig.1.5. In order to find the components of the vector $\vec{v}$ in the "red" frame of reference we must rotate the frame of reference by the angle ϕ.

Fig.1.5. The vector $\vec{v}$ in the "black" frame of reference is given by:

$$v_x + iv_y = ve^{i\varphi} \tag{1.18}$$

while the same vector $\vec{v}$ in the "red" frame of reference is given by:

$$v_r + iv_\varphi = ve^{i(\psi-\varphi)} \tag{1.19}$$

p_r and p_ϕ are the components parallel and perpendicular to the position vector $\vec{r}$ respectively, in other words they are radial and transversal components of the vector $\vec{r}$. This means that in order to get the radial and transversal components of velocity and acceleration we have to multiply equations (1.17) by exp(iϕ). As a result we get:

	Radial component	Transversal component
velocity	$\dot{r}$	$r\dot{\varphi}$
acceleration	$\ddot{r} - r\dot{\varphi}^2$	$2\dot{r}\dot{\varphi} + r\ddot{\varphi}$

1.3. Force and Motion, Newton's Second Law

When we push a physical body we apply the force of our body to move a body. We feel a certain strain in our body and we say we applied a force. In mechanics force is not meant (understood) a physiological feeling but the physical cause changing the state of motion of bodies. Forces result from interaction between bodies. The change of motion is equivalent to acceleration different from zero. As we know the relation between force and acceleration is given by the Newton's Second Law:

$$m\vec{a} = m\ddot{\vec{r}} = \vec{F} \tag{1.20}$$

m is the inertial mass of a body. Using the same device to accelerate bodies we obtain:

$$m_i\vec{a}_i = m_j\vec{a}_j = (m_i + m_j)\vec{a}_{i+j}$$

The inertial mass is an additive quantity, i.e.:

$$m_{1+2} = m_1 + m_2 \tag{1.21}$$

where m_{i+j} is the mass of a body consisting of two bodies of mass m_i and m_j kept together.

The equation (1.20) is equivalent to three scalar equations:

$$m\ddot{x} = F_x$$
$$m\ddot{y} = F_y \qquad (1.22)$$
$$m\ddot{z} = F_z$$

Taking into account the relation $\ddot{\vec{r}} = d\vec{v}/dt$ we obtain assuming that m=constant:

$$\frac{d\vec{p}}{dt} = \vec{F} \qquad (1.23)$$

Equation (1.23) is more general form of equation (1.20). Calculating the integral of (1.23) in respect to time we obtain:

$$\int_{t_0}^{t} \frac{d\vec{p}}{dt} dt = \vec{p} - \vec{p}_0 = \int_{t_0}^{t} \vec{F} dt \qquad (1.24)$$

The integral $\int_{t_0}^{t} \vec{F} dt$ is called the impulse of the force $\vec{F}$.

> *INSERT: When the impulse of a force is equal to zero, no change of momentum is observed. This leads to the Law of Conservation of Momentum. The impulse of a force can be equal to zero if either* $\vec{F} = 0$ *or the time during which a force is applied is equal to zero. In many practical cases the time of application of a force is so short that it can be assumed to be zero to the first approximation. This is the case of well known problem of a shell exploding at the highest point of its trajectory.*

1.3.1. Conservation of momentum

If the impulse of a force equals zero then we have for a body:

$$\vec{p} = \vec{p}_0 \qquad (1.25)$$

which just means that the momentum a body remains constant.

1.3.2. Conservation of energy, potential field

Let us assume that a point-like particle moves under the time-dependent force $\vec{F}$. The Newton's Law for such a body is of the form:

$$m\dot{\vec{v}} = \vec{F} \qquad (1.26)$$

Multiplying (1.26) by $\vec{v}$ and taking into account $\frac{d}{dt}\left(\vec{v}^2\right) = 2\vec{v}\dot{\vec{v}}$ we obtain:

$$\frac{d}{dt}\left(\frac{m\vec{v}^2}{2}\right) = \vec{F}\vec{v} \qquad (1.27)$$

Denoting the kinetic energy $\frac{m\vec{v}^2}{2} = T$ we get:

$$\frac{dT}{dt} = \vec{F}\vec{v} \qquad (1.28)$$

Calculating the integral of both sides of equation (1.28) we obtain:

$$\int_{t_0}^{t} \frac{dT}{dt} dt = T - T_0 = \int_{t_0}^{t} \vec{F}\vec{v} dt \tag{1.29}$$

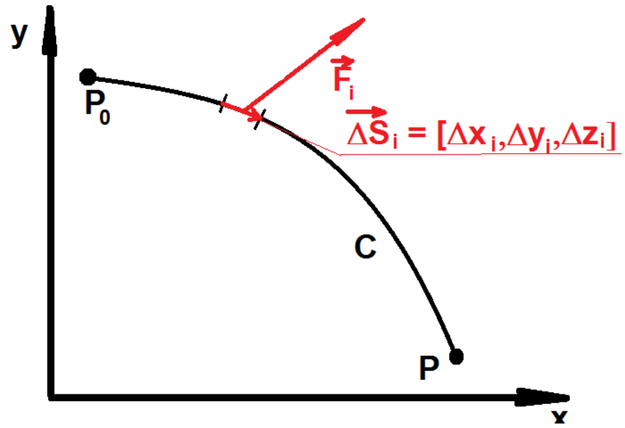

Fig.1.6. Point-like particle moves along the curve C under a time dependent force $\vec{F}(t)$.

Let us consider the right-side part of equation (1.29). Both the force and the velocity are vectors and may be time-dependent (see Fig.1.6). The integral can be rewritten in the form:

$$\int_{t_0}^{t} \vec{F}\vec{v} dt = \int_{t_0}^{t} \left(F_x v_x + F_y v_y + F_z v_z\right) dt = \lim_{n \Rightarrow \infty} \sum_{i=1}^{n} \left(F_{x_i} v_{x_i} + F_{y_i} v_{y_i} + F_{z_i} v_{z_i}\right) \Delta t_i =$$

$$\lim_{n \Rightarrow \infty} \sum_{i=1}^{n} \left(F_{x_i} \Delta x_i + F_{y_i} \Delta y_i + F_{z_i} \Delta z_i\right) = \int_{P_0}^{P} F_x dx + F_y dy + F_z dz = \int_{P_0}^{P} \vec{F} d\vec{s} \tag{1.30}$$

The expression $\int_{P_0}^{P} F_x dx + F_y dy + F_z dz = \int_{P_0}^{P} \vec{F} d\vec{s}$ is a curvilinear integral and is equal to the work of the force $\vec{F}$ at the curve C between the points P_0 and P.

1.3.2.1. Potential and conservative fields

Let us assume that there exist in a three-dimensional space such a scalar function $V(\vec{r}, t)$ that at any point of the space the force acting on a body is given by:

$$\vec{F} = -gradV(\vec{r}, t) \tag{1.31}$$

The function $V(\vec{r}, t)$ is called the potential of the field. If the potential is time independent the field is conservative. The work in a conservative field is equal to:

$$W = \int_{P_0}^{P} \vec{F} d\vec{s} = -\int_{P_0}^{P} gradV d\vec{s} = -\int_{P_0}^{P} \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \tag{1.32}$$

<u>MATHEMATICAL INSERT:</u> The expression $f_1(x, y, z)dx + f_2(x, y, z)dy + f_3(x, y, z)dz$ *is the total differential if there exists such a function U(x,y,z) that the following equations are fulfilled:*

$$\frac{\partial U}{\partial x} = f_1$$
$$\frac{\partial U}{\partial y} = f_2 \qquad (1.33)$$
$$\frac{\partial U}{\partial z} = f_3$$

The curvilinear integral of the total differential is given by:

$$\int_{P_0}^{P} tot.diff. = U(P) - U(P_0) \qquad (1.34)$$

Taking into account the above property of the integral we get for a conservative field:

$$W = \int \vec{F} d\vec{s} = -(V(\vec{r}) - V(\vec{r}_0)) = V(\vec{r}_0) - V(\vec{r}) \qquad (1.35)$$

so we get:

$$V(\vec{r}) = V(\vec{r}_0) - \int_{P_0}^{P} \vec{F} d\vec{s} \qquad (1.36)$$

It results from equation (1.36) that the potential is a relative quantity, i.e. in order to define potential of a point in space we have to define the potential of the reference point $\vec{r}_0$.

Combining ((1.36) and (1.26) we obtain the Law of Conservation of Energy in a conservative field:

$$V+T = V_0+T_0 = \text{constant} \qquad (1.37)$$

1.3.3. Conservation of Angular Momentum

Let us assume a point-like particle moves in a 3D space (see Fig.1.7). The equation of motion is:

$$\frac{d(m\vec{v})}{dt} = \vec{F} \qquad (1.38)$$

Multiplying (1.38) by $\vec{r}\times$ we get:

$$\vec{r} \times \frac{d(m\vec{v})}{dt} = \vec{r} \times \vec{F} \qquad (1.39)$$

Taking into account that $\vec{v} \times m\vec{v} = 0$ we obtain:

$$\frac{d}{dt}(\vec{r} \times m\vec{v}) = \vec{r} \times \vec{F}$$
$$\frac{d\vec{J}}{dt} = \vec{D} \qquad (1.40)$$

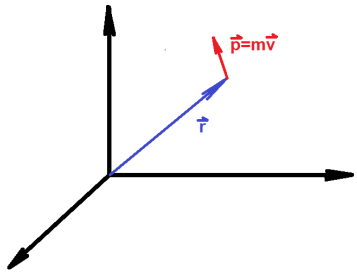

Fig.1.7. Motion of a point-like particle in 3D space.

where $\vec{J} = \vec{r} \times m\vec{v}$ is the angular momentum and $\vec{D} = \vec{r} \times \vec{F}$ id the moment of the force $\vec{F}$ about the zero point of the reference frame. As results from the definition the vector of angular momentum is perpendicular both to the position vector $\vec{r}$ and to the momentum $m\vec{v}$, i.e. the angular momentum is perpendicular to the plane of motion. If for some reasons the moment of force equals zero, the angular momentum is time-independent. In other words we have to do with a plane motion in such a case.

1.4. Central force

The central force is defined by the equation:

$$\vec{F} = \frac{\vec{r}}{r} F \tag{1.41}$$

where $\vec{r}$ is a position vector, r is its length so $\vec{r}/r$ is a unit vector parallel to the position vector (see Fig.1.8). The moment of a central force about the zero of the frame of reference

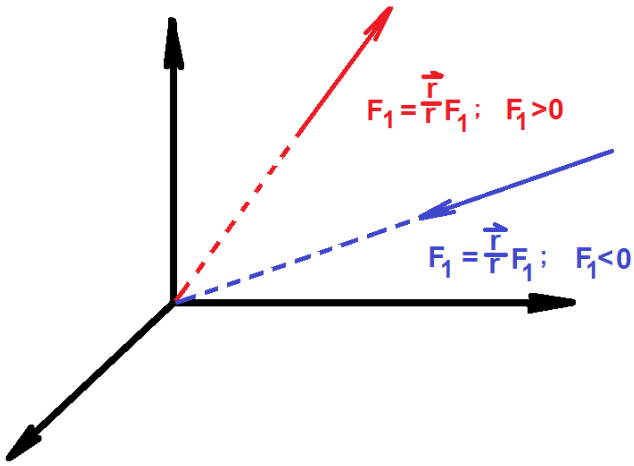

Fig.1.8. Definition of central force.

is equal to:

$$\vec{D} = \vec{r} \times \frac{\vec{r}}{r} F = 0 \tag{1.42}$$

so any motion in the field of central force is a plane phenomenon. We shall prove that if the potential of field of a force depends only on the length of the position vector V=V(r) the field of the force is a central one.

Assuming that we have to do with a conservative field and that V=V(r) we get the following expression for the force taking into account that V(r(x,y,z)) is a composition of functions :

$$\vec{F} = -gradV(r) = -\left(\vec{i}\,\frac{\partial V}{\partial x} + \vec{j}\,\frac{\partial V}{\partial y} + \vec{k}\,\frac{\partial V}{\partial z}\right) =$$
$$= \left(\vec{i}\,\frac{dV}{dr}\frac{\partial r}{\partial x} + \vec{j}\,\frac{dV}{dr}\frac{\partial y}{\partial y} + \vec{k}\,\frac{dV}{dr}\frac{\partial r}{\partial z}\right) =$$
$$-\frac{dV}{dr}\left(\vec{i}\,\frac{x}{\sqrt{x^2+y^2+z^2}} + \vec{j}\,\frac{y}{\sqrt{x^2+y^2+z^2}} + \vec{k}\,\frac{z}{\sqrt{x^2+y^2+z^2}}\right) =$$
$$= -\frac{dV}{dr}\frac{1}{r}\left(\vec{i}x + \vec{j}y + \vec{k}z\right) = -\frac{dV}{dr}\frac{\vec{r}}{r} \qquad (1.43)$$

1.4.1. Binet's formula

As shown above, in the case of motion in a field of central force the motion takes place in a plane. This enables to describe such a motion with polar coordinates. Our task is to find the equation of motion of a point-like particle in a field of central force using the polar coordinates, i.e. we want to find the equation of trajectory in the form r=r(ϕ).

The angular momentum is given by[1]:

$$J = \left|\vec{r} \times m\vec{v}\right| = mrv_{\perp} = mr^2\dot{\varphi} \qquad (1.44)$$

The equation of motion for a field of central force is in the form:

$$m\vec{a} = F_r\,\frac{\vec{r}}{r} \qquad /\bullet\frac{\vec{r}}{r} \qquad (1.45)$$

so we obtain:

$$ma_r = F_r \qquad (1.46)$$

where a_r is the radial acceleration and F_r is the central force (the zero of the frame of reference is in the centre of the field). Taking into account the formula for radial acceleration and (1.44) we get the set of two equations:

$$m\left(\ddot{r} - r\dot{\varphi}^2\right) = F_r$$
$$J = mr^2\dot{\varphi} \qquad (1.47)$$

In order to obtain trajectory of motion in the form r=r(ϕ) we have to eliminate time from the set of equations. The radius vector $\vec{r}$ can be represented by the following composition of functions r=r(ϕ(t)), so we obtain:

$$\dot{r} = \frac{dr}{d\phi}\dot{\phi} = \frac{dr}{d\phi}\frac{J}{mr^2} = -\frac{J}{m}\frac{d\left(\frac{1}{r}\right)}{d\phi} \qquad (1.48)$$

[1] Because the movement is a plane one, we can use the scalar value of the angular momentum as we know that the angular momentum is perpendicular to the plane of movement and it direction is constant.

$$\ddot{r}=\frac{d\dot{r}}{d\phi}\frac{d\phi}{dt}=\frac{d\dot{r}}{d\phi}\frac{J}{mr^2}=-\frac{J}{m}\frac{d^2\left(\frac{1}{r}\right)}{d\phi^2}\frac{J}{mr^2}=-\frac{J^2}{m^2r^2}\frac{d^2\left(\frac{1}{r}\right)}{d\phi^2} \qquad (1.49)$$

$$m\left(-\frac{J^2}{m^2r^2}\frac{d^2\left(\frac{1}{r}\right)}{d\phi^2}-\frac{J^2}{m^2r^3}\right)=F_r \qquad (1.50)$$

and finally we get

$$\frac{-J_2}{mr^2}\left\{\frac{d^2\left(\frac{1}{r}\right)}{d\phi^2}+\frac{1}{r}\right\}=F_r \qquad \textbf{\textit{Binet's formula}} \qquad (1.51)$$

Equation (1.51) is known as Binet's formula. Let us use the formula to solve the common case of field of central force in which the radial force is of the form $F_r=-k\frac{1}{r^2}$.

EXAMPLE: The central force given by $F_r=-k\frac{1}{r^2}$. *The Binet's equation is:*

$$-\frac{J^2}{mr^2}\left\{\frac{d^2\left(\frac{1}{r}\right)}{d\phi^2}+\frac{1}{r}\right\}=-k\frac{1}{r^2} \qquad \textit{so we get:}$$

$$\frac{d^2\left(\frac{1}{r}\right)}{d\phi^2}+\frac{1}{r}=\frac{km}{J^2} \qquad \textit{let's now substitute } \frac{1}{r}=x \textit{ and we get:}$$

$$\frac{d^2x}{d\phi^2}+x=\frac{km}{J^2}$$

function $x=A\cos\phi+\frac{km}{J^2}$ is a solution of the above equation, so we obtain**:**

$$r=\frac{\frac{J^2}{km}}{1+\frac{AJ^2}{km}\cos\phi}$$

The above equation can be recognized as the ellipse equation[2].

1.5. Constrained motion of a point-like particle

Let us suppose that a motion of a point-like particle is constrained (limited) to a surface of a sphere, the centre of the sphere is at the zero point of Cartesian coordinates (see Fig.1.9).

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[2] $r=\frac{p}{1+e\cdot\cos\phi}$. If e<1 we get the ellipse, if e=1 we get parabola, if e>1 we get hyperbola.

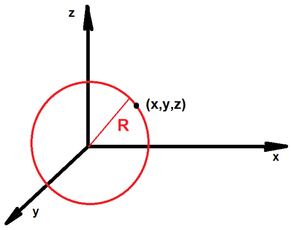

Fig.1.9. Point-like particle is constrained to the surface of a sphere.

The coordinates of the point-like particle have to satisfy the equation:

$$x^2 + y^2 + z^2 = R^2 \qquad (1.52)$$

The equation is called the equation of constraints.

EXAMPLE 1: Point-like particle moves on a surface of vertical cylinder. The radius of cylinder's base increases linearly with time (see Fig.1.10).

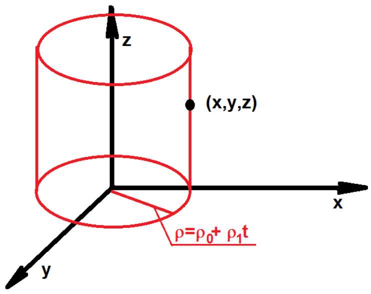

Fig.1.10. Particle on a surface of vertical cylinder.

The coordinates (x,y,z) of the point remaining on the surface of such a cylinder have to satisfy the equation $x^2 + y^2 = \rho_0 + \rho_1 t$.

EXAMPLE 2: A point-like particle moves inside of a sphere shown in Fig.1.9.

In this case the equation of constraints becomes inequality of the form:

$x^2 + y^2 + z^2 - R^2 < 0$ *or* $x^2 + y^2 + z^2 - R^2 \leq 0$ *depending on whether the points of the surface are available for the particle or not.*

EXAMPLE 3: A point-like particle moves at a circle in the xz plane (see Fig.1.11).

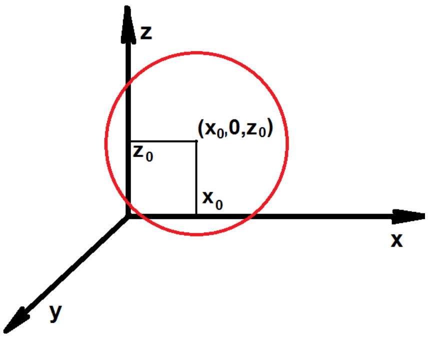

Fig.1.11. Particle at a circle in the xz plane.

Equations of constraints:

$$(x - x_0)^2 + (z - z_0)^2 - R^2 = 0$$
$$y = 0$$

EXAMPLE 4: Motion of a point-like particle is restricted to a surface of a sphere moving in space. The equation of constraints are

$$(x - at)^2 + (y - bt)^2 + (z - ct)^2 - R^2 = 0$$

In general a point-like particle or system of particles are not usually free to execute purely arbitrary motions. The motions are often required to satisfy certain geometrical conditions called constraints. If equations (or inequalities) of constraints can be written in the form

$$f(x,y,z,t)=0 \quad \text{or} \quad f(x,y,z,t)\leq 0 \quad \text{or} \quad f(x,y,z,t)\geq 0 \tag{1.53}$$

such constraints are HOLONOMIC CONSTRAINTS. In some cases the equations of constraints contain time derivatives of coordinates, so the equations of constraints are of the form:

$$f(x,y,z,\dot{x},\dot{y},\dot{z},t)=0 \quad \text{or} \quad f(x,y,z,\dot{x},\dot{y},\dot{z},t)\leq 0 \quad \text{or} \quad f(x,y,z,\dot{x},\dot{y},\dot{z},t)\geq 0 \tag{1.54}$$

This kind of constraints are non-holonomic constraints.

In cases where these geometrical conditions do not change with time, the constraints are said to be FIXED or SCLERONOMUS or STATIONARY. In cases where they depend on time, the constraints are said to be VARIABLE or RHEONOMUS or NON-STATIONARY.

The constraints expressed by an equality are BILATERAL CONSTRAINTS, the constraints expressed by an inequality are UNILATERAL CONSTRAINTS.

As we see, a constraint is a geometric or kinematic condition that limits the possibilities of motion. In such a case we say that a particle or a system of particles is subject to constraints given by equations (1.53) or (1.54).

1.5.1. Constrain forces, work of constrain forces

The equation of motion of a point-like particle restricted by some constraints reads:

$$m\ddot{\vec{r}}=\vec{F}+\vec{F}_R \tag{1.55}$$

where $\vec{F}$ is the applied force and $\vec{F}_R$ is the constrained (reaction) force. It results from all experiments and observations that ***the constrained force is perpendicular to the surface of constraints*** provided that ***friction is included in the applied forces***. If so, the constrain force for the case of motion on a surface given by equation f(x,y,z)=0 can be written as:

$$\vec{F}_R=\lambda\cdot grad(f) \tag{1.56}$$

If a motion of a point-like particle is restricted on a curve given by two equations $f_1(x,y,z)=0$

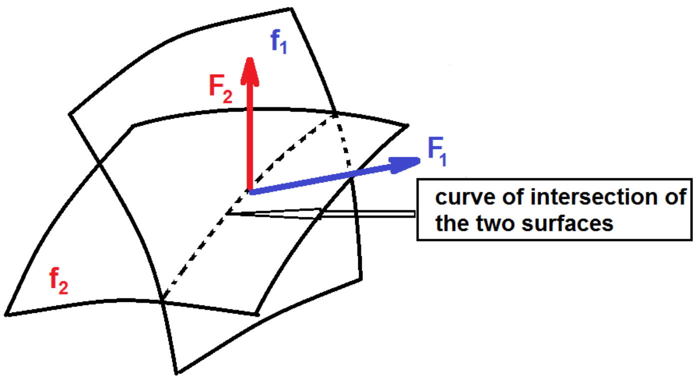

Fig.1.12. The reaction forces resulting from interaction with the two surfaces are perpendicular to the surfaces. The total reaction force is a linear combination of the two reaction forces F_1 and F_2 (see equation (1.57).

and $f_2(x,y,z)=0$ (see Fig.1.12) the reaction force is given by:

$$\vec{F}_R=\lambda_1\cdot grad(f_1)+\lambda_2\cdot grad(f_2) \tag{1.57}$$

1.5.2. *Work of reaction forces*

Work of a reaction force in case of motion at a surface defined as f(x,y,z,t)=0 is given by:

$$W_R = \vec{F}_R \cdot d\vec{s} = \lambda grad(f) \cdot \dot{\vec{r}} dt \tag{1.58}$$

In case of fixed constraints we have:

$$0 = \frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} = grad(f) \cdot \dot{\vec{r}} \tag{1.59}$$

Because the vector $\dot{\vec{r}}$ is perpendicular to grad(f), the work of reaction force in case of fixed constraints is equal to zero. When we have to do with variable constraints:

$$0 = \frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} + \frac{\partial f}{\partial t} = grad(f) \cdot \dot{\vec{r}} + \frac{\partial f}{\partial t} \tag{1.60}$$

Because the derivative $\partial f / \partial t$ need not be zero, the work of reaction forces is not equal to zero in this case.

1.5.3. *Motion at a surface – equation of motion*

Let us assume that a point-like particle moves at a surface f(x,y,z,t) under an applied force $\vec{F}$. The equations describing the motion are as follows:

$$\begin{aligned} m\ddot{\vec{r}} &= \vec{F} + \vec{F}_R = \vec{F} + \lambda grad(f) \\ f(x, y, z, t &= 0 \end{aligned} \tag{1.61}$$

Our aim is to eliminate λ (which can be function of time) from the above equations in order to obtain equation of motion in the form:

$$m\ddot{\vec{r}} = \vec{F} + function(f, \dot{\vec{r}}, \vec{r}) \tag{1.62}$$

The first derivative of f(x,y,z,t) is:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} + \frac{\partial f}{\partial t} = grad(f) \cdot \dot{\vec{r}} + \frac{\partial f}{\partial t} \tag{1.63}$$

and the second one is as follows:

$$\frac{d^2 f}{dt^2} = \ddot{\vec{r}} grad(f) + \dot{\vec{r}} \frac{d}{dt}(grad(f)) + \frac{d}{dt}\frac{\partial f}{\partial t} \tag{1.64}$$

Combining (1.64) and (1.61) we get:

$$m\ddot{\vec{r}} = \vec{F} + \lambda grad(f) \tag{1.65}$$

where λ is given by:

$$\lambda = \frac{-m\left(\dot{\vec{r}} \frac{d}{dt} grad(f) + \frac{d}{dt}\frac{\partial f}{\partial t}\right) - \vec{F} grad(f)}{(grad(f))^2} \tag{1.66}$$

The above equations are quite complex and though it is often possible to solve them in many practical cases using contemporary computer numerical methods, there exist better methods to solve motion of constrained systems. The methods will be a subject of the future lectures.

1.5.4. Motion at a curve – equation of motion

In case of motion at a curve defined as an intersection of two surfaces $f_1(x,y,z,t)$ and $f_2(x,y,z,t)$ the set of equations of motion reads:

$$
\begin{aligned}
& m\ddot{\vec{r}} = \vec{F} + \lambda_1 grad(f_1) + \lambda_2 grad(f_2) \\
& f_1(x,y,z,t) = 0 \\
& f_2(x,y,z,t) = 0
\end{aligned}
\tag{1.67}
$$

It is possible to eliminate λ_1 and λ_2 from the above set of equations in order to get the equation in the form:

$$m\ddot{\vec{r}} = \vec{F} + function(\dot{\vec{r}}, \vec{r}, f_1, f_2) \tag{1.68}$$

but the equations obtained are so complex it does not pay to solve the problem in this way. However, it often is possible to get much useful information about a motion in the way presented below. We shall also get familiar with the so-called Γ special function.

Let us multiply the equation of motion by a unity vector $\vec{t}$ tangent to the trajectory:

$$
\begin{aligned}
& m\vec{a} \cdot \vec{t} = \vec{F} \cdot \vec{t} + \vec{F}_R \cdot \vec{t} \\
& m\vec{a}_t = F_t + 0 \\
& m\vec{a}_t = F_t
\end{aligned}
\tag{1.69}
$$

where a_t is a tangent acceleration, F_t is a tangent component of a force applied. $\vec{F}_R \cdot \vec{t} = 0$ because the reaction force is perpendicular to the trajectory. As a consequence we get:

$$m\ddot{s} = F_t \tag{1.70}$$

s is the distance covered by a particle, the distance is equal to the length of curve between the initial and the final points of motion. Let us use the above equation to solve a problem of mathematical pendulum for the amplitude angle equal to $\pi/2$ (see Fig.1.13).

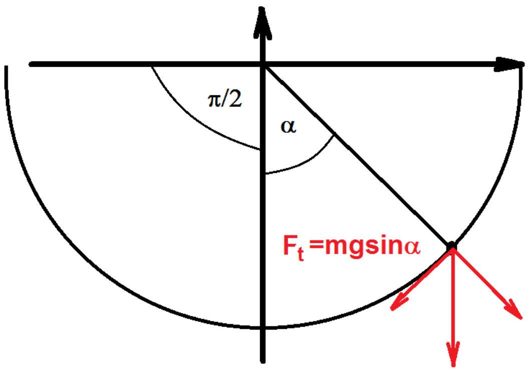

Fig.1.13. Mathematical pendulum. The amplitude of motion is equal to π/2.

The tangent component of force is given by:

$$F_t = -mg \sin\alpha \tag{1.71}$$

So we get:

$$m\ddot{s} = -mg \sin\alpha \tag{1.72}$$

Using simple geometrical relations $s = l\alpha$ and $\ddot{s} = l\ddot{\alpha}$ we obtain:

$$\ddot{\alpha} = -\frac{g}{l}\sin\alpha \qquad \text{multiplying by } /\cdot\dot{\alpha} \tag{1.73}$$

$$\ddot{\alpha}\dot{\alpha} = -\frac{g}{l}\dot{\alpha}\sin\alpha \tag{1.74}$$

Taking into account that $\frac{1}{2}\frac{d}{dt}\left(\dot{\alpha}^2\right) = \dot{\alpha}\ddot{\alpha}$ and $\frac{d(\cos\alpha)}{dt} = -\sin\alpha\cdot\dot{\alpha}$ we get:

$$\frac{1}{2}\frac{d}{dt}\left(\dot{\alpha}^2\right) = -\frac{g}{l}\frac{d(\cos\alpha)}{dt} \tag{1.75}$$

Integrating the above equation we obtain:

$$\frac{1}{2}\dot{\alpha}^2 = \frac{g}{l}\cos\alpha + C \tag{1.76}$$

C is a constant. Because $\dot{\alpha} = 0$ when α=0 the constant C=0, so we get:

$$\frac{1}{2}\dot{\alpha}^2 = \frac{g}{l}\cos\alpha \tag{1.77}$$

Rearranging the equation we obtain:

$$\frac{d\alpha}{dt} = \sqrt{\frac{2g}{l}}\sqrt{\cos\alpha} \tag{1.78}$$

Separating the variables in the above equation we get:

$$\frac{d\alpha}{\sqrt{\cos\alpha}} = \sqrt{\frac{2g}{l}}dt \tag{1.79}$$

Integrating the above equation between the angles 0 and π/2 we get:

$$\int_0^{\pi/2}\frac{d\alpha}{\sqrt{\cos\alpha}} = \sqrt{\frac{2g}{l}}\int_0^{T/4}dt = \sqrt{\frac{2g}{l}}\frac{T}{4} \tag{1.80}$$

The integral at the left side of the above equation cannot be expressed by algebraic or trigonometric functions. In order to calculate the left-side integral we have to use co-called Γ-function.

1.5.4.1. Gamma function and Beta function

The definition of Gamma function is as follows:

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx \tag{1.81}$$

Let us prove the following theorem:

$$\Gamma(p+1) = p\Gamma(p) \tag{1.82}$$

Using the definition (1.81) we obtain $\Gamma(p+1) = \int_0^\infty x^p e^{-x}dx$.

Calculating the integral by integration by parts we get:

$$\Gamma(p+1) = -x^p e^{-x}\Big|_0^\infty - p\int_0^\infty x^{p-1}\left(-e^{-x}\right)dx = p\int x^{p-1}e^{-x}dx = p\Gamma(p) \qquad (1.83)$$

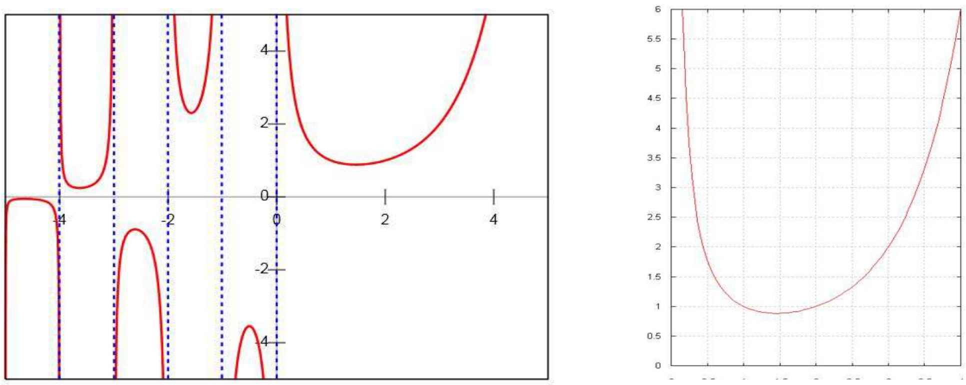

Fig.1.14. Gamma function in the range between (-5,5) and (0,4)[3].

For p=1 $\Gamma(1) = \int_0^{\pi/2} e^{-x}dx = 1$, for p=2 Γ(2)=1·Γ(1)=1, for p=3 Γ(3)=2· 1=2, and in general:

$$\Gamma(n+1)=n! \qquad (1.84)$$

The shape of Gamma function in the range of p (-5,5) is shown in Fig.1.14.

The definition of Beta function is as follows:

$$\beta(p,q) = 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}d\theta \qquad (1.85)$$

The relation between Gamma function and Beta function is given by:

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad (1.86)$$

For p=1/2 and q=1/4 Beta function is equal to:

$$\beta\left(\frac{1}{2},\frac{1}{4}\right) = \int_0^{\pi/2} (\cos\alpha)^{\frac{-1}{2}}d\alpha = \frac{T}{4}\sqrt{\frac{2g}{l}} \qquad (1.87)$$

So we obtain:

$$\frac{T}{4}\sqrt{\frac{2g}{l}} = \beta\left(\frac{1}{2},\frac{1}{4}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \qquad (1.88)$$

Taking into account that:

[3] Wikipedia

$$\Gamma\left(\frac{1}{2}\right)=1.7724538509$$

$$\Gamma\left(\frac{1}{4}\right)=3.6256099082$$

$$\Gamma\left(\frac{3}{4}\right)=1.2254167024$$

we obtain the following relation:

$$T \cong 7.4163\sqrt{\frac{l}{g}} \qquad (1.89)$$

1.6. D'Alembert principle

D'Alembert principle is another form of equations of motion, very useful for our further considerations. We shall consider three various cases of motion of a point-like particle.

1.6.1. Free point-like particle

D'Alembert principle for a free point-like particle reads:

$$\left(\vec{F}-m\ddot{\vec{r}}\right)\delta\vec{r}=0 \qquad (1.90)$$

where $\delta\vec{r}$ is an arbitrary vector. We shall prove that such a form of equation of motion is equivalent to the Newton's second law:

$$\vec{F}=m\ddot{\vec{r}} \qquad (1.91)$$

I: If $\vec{F}=m\ddot{\vec{r}}$ then $\vec{F}-m\ddot{\vec{r}}=0$ so multiplying by arbitrary vector $\delta\vec{r}$ we obtain $(\vec{F}-m\ddot{\vec{r}})\delta\vec{r}=0$.

II: If $\left(\vec{F}-m\ddot{\vec{r}}\right)\delta\vec{r}=0$ for arbitrary $\delta\vec{r}$, this can be true only if $\vec{F}-m\ddot{\vec{r}}=0$ which leads to the Newton's second law.

1.6.2. Point-like particle on a surface

The Newton's equation of motion in this case is of the form:

$$\begin{aligned} &m\ddot{\vec{r}}=\vec{F}+\lambda grad(f) \\ &f(\vec{r},t)=0 \end{aligned} \qquad (1.92)$$

We shall show that the above equations are equivalent to d'Alembert principle for such a case given in the form:

$$\begin{aligned} &\left(\vec{F}-m\ddot{\vec{r}}\right)\delta\vec{r}=0 \\ &f(\vec{r},t)=0 \\ &grad(f)\cdot\delta\vec{r}=0 \end{aligned} \qquad (1.93)$$

In this case the vector $\delta\vec{r}$ is not arbitrary, it satisfies the additional condition $grad(f)\cdot\delta\vec{r}=0$.

I: If we multiply the equation $m\ddot{\vec{r}}=\vec{F}+\lambda grad(f)$ by $\delta\vec{r}$ satisfying the condition $grad(f)\cdot\delta\vec{r}=0$ we obtain immediately (1.93)

II: Let us multiply the third equation of (1.93) by an arbitrary (for the time being) λ and let us add the result to the first equation of (1.93). We get:

$$\left(\vec{F} + \lambda grad(f) - m\ddot{\vec{r}}\right)\delta\vec{r} = 0 \qquad (1.94)$$

Now let us analyse the condition $grad(f)\cdot\delta\vec{r} = 0$. It can be rewritten in the form:

$$\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z = 0 \qquad (1.95)$$

Vector grad(f) cannot be equal to zero which means that at least one of its components is different from zero. Let us assume that $\frac{\partial f}{\partial x} \neq 0$. The x-component of $\delta\vec{r}$ can be written as:

$$\delta x = \frac{-\left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z\right)}{\frac{\partial f}{\partial x}} \qquad (1.96)$$

It results from (1.96) that the components δy and δz remain independent and arbitrary, but the x-component δx does depend on the other two. Let us rewrite (1.94) in the form:

$$\left(X + \lambda\frac{\partial f}{\partial x} - m\ddot{x}\right)\delta x + \left(Y + \lambda\frac{\partial f}{\partial y} - m\ddot{y}\right)\delta y + \left(Z + \lambda\frac{\partial f}{\partial z} - m\ddot{z}\right)\delta z = 0 \qquad (1.97)$$

X,Y and Z are components of the force $\vec{F}$. The coefficient λ has been assumed to be arbitrary so far. Let us take such a value of λ for which the following expression is satisfied:

$$X + \lambda\frac{\partial f}{\partial x} - m\ddot{x} = 0 \qquad (1.98)$$

This just means that the λ is given by $\lambda = \frac{m\ddot{x} - X}{\frac{\partial f}{\partial x}}$. It is possible because we assumed that $\frac{\partial f}{\partial x} \neq 0$. If so, (1.97) is reduced to the form:

$$\left(Y + \lambda\frac{\partial f}{\partial y} - m\ddot{y}\right)\delta y + \left(Z + \lambda\frac{\partial f}{\partial z} - m\ddot{z}\right)\delta z = 0 \qquad (1.99)$$

The above equation has to be satisfied for arbitrary values of δy and δz. This is possible only when

$$Y + \lambda\frac{\partial f}{\partial y} - m\ddot{y} = 0 \qquad (1.100)$$

and

$$Z + \lambda\frac{\partial f}{\partial z} - m\ddot{z} = 0 \qquad (1.101)$$

Equations (1.98), (1.100) and (1.101) are equivalent to (1.92).

1.6.3. Point-like particle at a curve

The equations of motion are as follows:

$$m\ddot{\vec{r}} = \vec{F} + \lambda_1 grad(f_1) + \lambda_2 grad(f_2)$$
$$f_1(\vec{r},t) = 0 \qquad (1.102)$$
$$f_2(\vec{r},t) = 0$$

$f_1(\vec{r},t)=0$ and $f_2(\vec{r},t)=0$ are equations of two surfaces intersecting along a curve defined in this way. The two functions are independent of each other to avoid two parallel surfaces which do not intersect. We shall prove that the equations (1.102) are equivalent to d'Alembert principle written in the form:

$$(\vec{F} - m\ddot{\vec{r}})\delta\vec{r} = 0$$
$$f_1(\vec{r},t) = 0$$
$$f_2(\vec{r},t) = 0 \qquad (1.103)$$
$$grad(f_1)\cdot\delta\vec{r} = 0$$
$$grad(f_2)\cdot\delta\vec{r} = 0$$

I: If we multiply the first equation of (1.102) by $\delta\vec{r}$ satisfying the conditions $grad(f_1)\cdot\delta\vec{r}=0$ and $grad(f_2)\cdot\delta\vec{r}=0$ we get the first equation of (1.103).

II: Let us multiply the last two equations of (1.103) by arbitrary (for the time being) coefficients λ_1 and λ_2 and let us add them to the first one of (1.103). We get:

$$(\vec{F} + \lambda_1 grad(f_1) + \lambda_2 grad(f_2) - m\ddot{\vec{r}})\delta\vec{r} = 0 \qquad (1.104)$$

Equation (1.104) can be written in the form:

$$\left(X + \lambda_1 \frac{\partial f_1}{\partial x} + \lambda_2 \frac{\partial f_2}{\partial x} - m\ddot{x} \right)\delta x + \left(Y + \lambda_1 \frac{\partial f_1}{\partial y} + \lambda_2 \frac{\partial f_2}{\partial y} - m\ddot{y} \right)\delta y + \qquad (1.105)$$
$$+\left(Z + \lambda_1 \frac{\partial f_1}{\partial z} + \lambda_2 \frac{\partial f_2}{\partial z} - m\ddot{z} \right)\delta z = 0$$

MATHEMATICAL INSERT:

Dependent and independent functions

Let us consider a set of m functions of n variables $f_1(x_1,\ldots,x_n),\ldots, f_m(x_1,\ldots,x_n)$. Let us assume that values of one of the functions $f_j(x_1,\ldots,x_n)$ is uniquely specified by the other functions:

$$f_j(x_1,\ldots,x_n)=\phi(f_1,\ldots,f_{j-1},f_{j+1},\ldots,f_m)$$

The function f_j is dependent on the other functions. If none of the functions composing the set of functions is dependent on the others, the functions are independent.

EXAMPLE: For the functions:

$$f_1(x_1,\ldots,x_n) = x_1x_2 - x_3$$
$$f_2(x_1,\ldots,x_n) = x_1x_3 + x_2$$
$$f_3(x_1,\ldots,x_n) = (x_1^2+1)(x_2^2+x_3^2) - (x_1^2-1)x_2x_3 - x_1(x_2^2-x_3)^2$$

The following identity is satisfied:

$$f_3 = f_1^2 - f_1 f_2 + f_2^2$$

***System of independent functions*.**

Let us assume that there exist m functions of n variables $f_1(x_1,\dots,x_n)$, …, $f_m(x_1,\dots,x_n)$ and n>m. If there exists a different from zero determinant of m degree in the Jacobi matrix

$$\begin{Bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{Bmatrix} \qquad (1.106)$$

the system of m functions is independent[4]*. The convert theorem is also true. Let us note that the system of two functions $f_1(x,y,z)$ and $f_2(x,y,z)$ is to be independent to define a curve.*

The two last equations of (1.103) are conditions which limit the values of the vector $\delta\vec{r}$. Let write them in the form:

$$\begin{aligned} \frac{\partial f_1}{\partial x}\delta x + \frac{\partial f_1}{\partial y}\delta y + \frac{\partial f_1}{\partial z}\delta z = 0 \\ \frac{\partial f_2}{\partial x}\delta x + \frac{\partial f_2}{\partial y}\delta y + \frac{\partial f_2}{\partial z}\delta z = 0 \end{aligned} \qquad (1.107)$$

The matrix of coefficients of the above set of equations is:

$$\begin{Bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{Bmatrix} \qquad (1.108)$$

Functions f_1 and f_2 are independent. This implies that it is possible to extract from the matrix determinant different from zero. Let us assume that the non-zero determinat is:

$$\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} \neq 0 \qquad (1.109)$$

If so, the following set of equations can be solved:

$$\begin{aligned} \frac{\partial f_1}{\partial x}\delta x + \frac{\partial f_1}{\partial y}\delta y = -\frac{\partial f_1}{\partial z}\delta z \\ \frac{\partial f_2}{\partial x}\delta x + \frac{\partial f_2}{\partial y}\delta y = -\frac{\partial f_2}{\partial z}\delta z \end{aligned} \qquad (1.110)$$

The solution of the above set of equation can be written in general as:

---

[4] The necessary and sufficient condition for m functions of n variables $f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n)$ to be independent is that it is possible to find a different from zero determinant of m degree in the Jacobi matrix of the system.

δx=function(δz)
δy=function(δz)
i.e. both δx and δy are dependent on δz. The only independent component of the vector $\delta\vec{r}$ is δz.

We choose such values of λ_1 and λ_2 in the first two components of (1.105) so that the following two equations are satisfied:

$$\lambda_1 \frac{\partial f_1}{\partial x} + \lambda_2 \frac{\partial f_2}{\partial x} = m\ddot{x} - X$$
$$\lambda_1 \frac{\partial f_1}{\partial y} + \lambda_2 \frac{\partial f_2}{\partial y} = m\ddot{y} - Y \qquad (1.111)$$

It is possible because the determinant of the above set of equations is determinat of inverse of the matrix (1.109) which is not equal to zero. So it remains from (1.105):

$$\left(Z + \lambda_1 \frac{\partial f_1}{\partial z} + \lambda_2 \frac{\partial f_2}{\partial z} - m\ddot{z} \right)\delta z = 0 \qquad (1.112)$$

for arbitrary values of the component δz. It can be possible only when the expression in the bracket is zero, so finally we obtain:

$$m\ddot{x} = X + \lambda_1 \frac{\partial f_1}{\partial x} + \lambda_2 \frac{\partial f_2}{\partial x}$$
$$m\ddot{y} = Y + \lambda_1 \frac{\partial f_1}{\partial y} + \lambda_2 \frac{\partial f_2}{\partial y} \qquad (1.113)$$
$$m\ddot{z} = Z + \lambda_1 \frac{\partial f_1}{\partial z} + \lambda_2 \frac{\partial f_2}{\partial z}$$

The above set of equations is equivalent to the equation (1.102).

1.7. Displacements real, possible and virtual

Real displacement results from solution of equations of motion for a given case. It is given by:

$$d\vec{r} = \dot{\vec{r}}dt \qquad (1.112)$$

Possible displacement is a displacement permissible (admissible) by constraints. For variable constraints a possible displacement $\Delta\vec{r}$ is given by:

$$\Delta f = 0 = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z + \frac{\partial f}{\partial t}\Delta t = grad(f)\cdot\Delta\vec{r} + \frac{\partial f}{\partial t}\Delta t \qquad (1.113)$$

For fixed constraints the possible displacement is:

$$0 = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z = grad(f)\cdot\Delta\vec{r} \qquad (1.114)$$

A real displacement is one of the possible displacements.

The most important for our further consideration is the definition of virtual displacement $\delta\vec{r}$. It is defined as:

$$grad(f)\cdot\delta\vec{r} = 0 \qquad (1.115)$$

According to the above definition virtual displacement is perpendicular to grad f, i.e. tangent to the surface of constraints, but in case of variable constraints the constrains are assumed to be stopped for a moment.

1.8. Constrained system of point-like particles, virtual displacement of constrained system

Let us consider the limits of motion of two point-like particles connected with a stiff rod. If the length of this rod is *l*, the equation describing the constrains are:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - l^2 = 0 \qquad (1.116)$$

If the two points were connected with a thread, the constraints would be unilateral and would be described by:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - l^2 \le 0 \qquad (1.117)$$

EXAMPLE 1: Two point-like particles are connected with a stiff rod and move on the xy plane at a circle. The length of the rod is l, the radius of the circle is R, its centre is at the zero of frame of reference.

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - l^2 = 0$$

$$x_1^{\ 2} + y_1^{\ 2} + z_1^{\ 2} - R^2 = 0$$

$$x_2^{\ 2} + y_2^{\ 2} + z_2^{\ 2} - R^2 = 0$$

$$z_1 = 0$$

$$z_2 = 0$$

EXAMPLE 2: Two point-like particles are connected with a stiff rod. Point O of this rod divides it into two segments in relation 2:1 and is fixed at the zero point of Cartesian frame of reference. The rod can rotate freely around point O.

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - l^2 = 0$$

$$x_1^{\ 2} + y_1^{\ 2} + z_1^{\ 2} - \left(\frac{l}{3}\right)^2 = 0$$

$$x_2^{\ 2} + y_2^{\ 2} + z_2^{\ 2} - \left(\frac{2l}{3}\right)^2 = 0$$

In general equations of constrains for a system of n point-like particles are:

$$f_k(\vec{r}_1, \vec{r}_2, ...\vec{r}_n, t) = 0 \qquad (1.118)$$

for bilateral constraints and:

$$\phi_k(\vec{r}_1, \vec{r}_2, ...\vec{r}_n, t) \le 0 \qquad (1.119)$$

for unilateral constraints. k=1,…,p is a number of equations (or inequalities).

1.9. Degrees of freedom

The number of independent coordinates necessary to define position of a free single point-like particle is 3. When motion of such a particle is restricted by 1 equation of constraints (for instance $x^2+y^2+z^2-R^2=0$) then one of the three coordinates can be calculated if the other two are known, so the number of coordinates necessary to define position of a particle is reduced by one.

For two free point-like particles the number of coordinates necessary to define position of the system is 6 and in general, the number of coordinates necessary to define position of a system of n free point-like particles is 3n. Each equation of constraints reduces the number of coordinates by one, so the number of degrees of freedom is given by:

$$f=3n-p \quad (1.120)$$

where p is the number of equations of constraints.

1.10. Virtual displacement of a system of point-like particles

For 2 point-like particles moving at a surface $f(\vec{r}_1,\vec{r}_2,t)=0$ their virtual displacement $\delta\vec{r}_1$ and $\delta\vec{r}_2$ is defined as (see Fig.1.15):

$$grad_1(f)\delta\vec{r}_1+grad_2(f)\delta\vec{r}_2=0 \quad (1.121)$$

where grad$_i$ is defined as:

$$grad_i=\vec{i}\frac{\partial}{\partial x_i}+\vec{j}\frac{\partial}{\partial y_i}+\vec{k}\frac{\partial}{\partial z_i} \quad (1.121a)$$

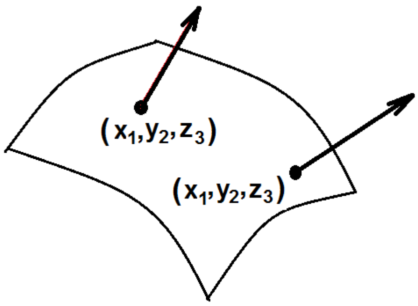

Fig.1.15. Gradients at two different points are perpendicular to the surface $f(\vec{r}_1,\vec{r}_2,t)=0$. The virtual displacements $\delta\vec{r}_1$ and $\delta\vec{r}_2$ have to be perpendicular to the gradients, i.e. they are tangent to the surface.

For motion at a curve defined by equations of two surfaces $f_1(\vec{r}_1,\vec{r}_2,t)=0$ and $f_2(\vec{r}_1,\vec{r}_2,t)=0$ the virtual displacement is defined as:

$$\begin{aligned}grad_1(f_1)\delta\vec{r}_1+grad_2(f_1)\delta\vec{r}_2=0\\ grad_1(f_2)\delta\vec{r}_1+grad_2(f_2)\delta\vec{r}_2=0\end{aligned} \quad (1.122)$$

It results from the above equations that the two components of the virtual displacement $\delta\vec{r}_1$ and $\delta\vec{r}_2$ are perpendicular to both surfaces with respect to coordinates of the two points.

For n point-like particles restricted by constraints given by p equations $f_k(\vec{r}_1,...\vec{r}_n,t)=0$, k=1,…,p, the virtual displacement is defined by the following p equations:

$$\sum_{i=1}^{n} grad_i(f_k)\cdot\delta\vec{r}_i=0 \qquad \text{for k=1,…,p} \quad (1.123)$$

1.11. Configuration space

Let us assume we have a system of n point-like particles, their position in space is defined by n radius-vectors $\vec{r}_i = [x_i, y_i, z_i]$, i=1,…,n. Let us define the following relation between the coordinates $[x_i,y_i,z_i]$ and new coordinates of a 3n-dimensional space corresponding to the previous ones:

$$\begin{aligned} x_1 &\Rightarrow x_1 \\ y_1 &\Rightarrow x_2 \\ z_1 &\Rightarrow x_3 \\ x_2 &\Rightarrow x_4 \\ &\ldots\ldots\ldots\ldots \\ x_n &\Rightarrow x_{3n-2} \\ y_n &\Rightarrow x_{3n-1} \\ z_n &\Rightarrow x_{3n} \end{aligned} \tag{1.124}$$

which can be written in general form:

$$x_i, y_i, z_i \Rightarrow x_{3i-2}, x_{3i-1}, x_{3i} \tag{1.125}$$

A point of 3n-dimensional space $x=[x_1,x_2,\ldots,x_{3n}]$ is defined in this way. This point represents position of the whole system and is often called a representative point.

An identical transformation of coordinates of forces holds:

$$X_i, Y_i, Z_i \Rightarrow X_{3i-2}, X_{3i-1}, X_{3i} \tag{1.126}$$

The new 3n-dimensional space is called a configuration space. In order to make easier to write equations of motions in a configuration space the following correspondence between mass of the i-point and the mass corresponding to consecutive coordinates of configuration space:

$$m_i \Rightarrow m_{3i-2} = m_{3i-1} = m_{3i} \tag{1.127}$$

Having defined the configuration space we can write down the position, the equations of motion and the virtual displacement in the way shown in the table below.

Position	$\vec{r}_i = [x_i, y_i, z_i]$ $i = 1,\ldots,n$	$x=[x_1,x_2,\ldots,x_{3n}]$; x represents all components of radius vector
Equations of motion	$m_i \ddot{\vec{r}}_i = \vec{F}_i$ $i = 1,\ldots,n$	$m_j \ddot{x}_j = X_j$ $j = 1,\ldots,3n$
Constraints	$f_k(\vec{r}_1,\ldots,\vec{r}_n,t) = 0$ $k = 1,\ldots,p$ f=3n-p	$f_f(x,t) = 0$ $k = 1,\ldots,p$ f=3n-p
Virtual displacement of a free system of point-like part.	$\delta\vec{r}_i = [\delta x_i, \delta y_i, \delta z_i]$ $i = 1,\ldots,n$ $\delta x_i,\delta y_i,\delta z_i$ are unrestricted	$\delta x=[\delta x_1,\delta x_2,\ldots,\delta x_{3n}]$ The values of δx_j are unrestricted, j=1,…,3n
Virtual displacement of a constrained system of point-like particles	$\sum_{i=1}^{n} grad_i(f_k)\cdot\delta\vec{r}_i = 0$ i=1,…,n	$\sum_{j=1}^{3n} \frac{\partial f_k}{\partial x_j}\delta x_j = 0$ k=1,…,p

1.12. Laws of motion of constrained systems

For a single point-like particle the equations of motion can be written in the form:

1. $CONTRAINTS \Rightarrow REACTION\_FORCES \quad \vec{F}_R$
2. $m\ddot{\vec{r}} = \vec{F} + \vec{F}_R$ where the reaction force $\vec{F}_R = \lambda grad(f)$
3. Definition of the virtual displacement $grad(f) \cdot \delta\vec{r} = 0$ leads to the conclusion that the work on the virtual displacement is $W = \vec{F}_R \cdot \delta\vec{r} = 0$.

For a system of point-like particles restricted by a number of equations of constraints the equations of motion are as follows:

1. $CONTRAINTS \Rightarrow REACTION\_FORCES \quad \overline{F}_{R_i}$ or in the configuration space X_{R_j}
2. $m_i\ddot{\vec{r}}_i = \vec{F}_i + \vec{F}_{R_i}$ or in the configuration space $m_j\ddot{x}_j = X_j + X_{R_j}$
3. The total work of reaction forces on the virtual displacements is equal to zero:

$$\sum_1^n \vec{F}_{R_i} \cdot \delta\vec{r}_i = \sum_{j=1}^{3n} X_{R_j}\delta x_j = 0 \qquad (1.128)$$

The first two points are quite obvious, the second one just results from the Newton's Second Law. The third point does not result from the Newton's Laws directly and cannot be derived from the Newton's Laws. It has been tested for numerous constrained systems and the final results justify acceptance of this point.

1.13. D'Alembert principle of a system in configuration space

Equations of motion of a system of n point-like particles can be written in the form:

$$\begin{aligned} & m_j\ddot{x}_j = X_j + X_{R_j} \\ & f_k(x,t) = 0 \qquad \text{for } k=1,\dots,p \\ & \sum_{j=1}^{3n} X_{R_j}\delta x_j = 0 \end{aligned} \qquad (1.129)$$

Let us multiply the first equation by the components of the virtual displacement defined by $\sum_{j=1}^{3n} \frac{\partial f_k}{\partial x_j}\delta x_j = 0$ (the table below) and let us summarise the results with respect to the index j.

$$\sum_{j=1}^{3n} m_j\ddot{x}_j\delta x_j = \sum_{j=1}^{3n} X_j\delta x_j + \sum_{j=1}^{3n} X_{R_j}\delta x_j \qquad (1.130)$$

The last component of the above equation is zero (the work on the virtual displacement), so we get:

$$\begin{aligned} & \sum_{j=1}^{3n} \left(X_j - m_j\ddot{x}_j\right)\delta x_j = 0 \\ & f_k(x,t) = 0 \qquad k=1,\dots,p \\ & \sum \frac{\partial f_k}{\partial x_j}\delta x_j = 0 \end{aligned} \qquad (1.131)$$

The above set of equations is ***d'Alembert principle***. The principle states that the sum of virtual works of applied and inertial forces, acting on a system subject to constraints given by a set of equations $f_k(x,t)=0$, is zero (k=1,…,p). The principle will be our starting point to obtain Lagrange equations of the first kind and the second kind.

1.14. Principle of virtual work

Let us suppose that a system of n point-like particles is in equilibrium. This implies that both $\dot{x}_j = \ddot{x}_j = 0$ for all coordinates. In consequence of that the d'Alembert principle becomes:

$$\begin{aligned} &\sum_{j=1}^{3n} X_j \delta x_j = 0 \\ &f_k(x,t) = 0 \qquad \text{for } k=1,\dots,p \\ &\sum \frac{\partial f_k}{\partial x_j} \delta x_j = 0 \end{aligned} \tag{1.132}$$

A system is in equilibrium if the virtual work of applied forces is zero.

> *The necessary and sufficient condition for static equilibrium of a system subject to fixed constraints is that the virtual work of the applied forces, for virtual displacements consistent with the constraints, be zero.*

2. LAGRANGE EQUATIONS

…..

2.1. Lagrange equations of the first kind

Let us assume we have a system of n point-like particles, their motion is restricted by p equations of constraints. The equations of motion of such a system in a configuration space can be written as:

$$m_j \ddot{x}_j = X_j + X_{R_j} \qquad \text{for } k=1,\ldots,p \qquad (1.133)$$
$$f_k(x,t) = 0$$

It may be supposed that the constraint forces X_{Rj} are dependent on the equations of constraints $f_k(x,t)$. Let us try to express the constrained forces acting on the particles by the functions $f_k(x,t)$. We shall use the method of ***Lagrange multipliers***. Let us start from the d'Alembert principle for a system of n point-like particles:

$$\sum_{j=1}^{3n} \left(X_j - m_j \right) \delta x_j = 0$$
$$f_k(x,t) = 0$$
$$\sum_{j=1}^{3n} \frac{\partial f_k}{\partial x_j} \delta x_j = 0$$

Let us multiply each of the last equations by arbitrary for the time being coefficient λ_k, then perform the summation over the index k:

$$\sum_{k=1}^{p} \lambda_k \sum_{j=1}^{3n} \frac{\partial f_k}{\partial x_j} \delta x_j = \sum_{j=1}^{3n} \sum_{k=1}^{p} \lambda_k \frac{\partial f_k}{\partial x_j} \delta x_j = 0 \qquad (1.134)$$

Now let us add the result to the first equation of d'Alembert principle. We obtain:

$$\sum_{j=1}^{3n} \left(X_j + \sum_{k=1}^{p} \lambda_k \frac{\partial f_k}{\partial x_j} - m_j \ddot{x}_j \right) \delta x_j = 0 \qquad (1.135)$$

The p equations $\sum_{j=1}^{3n} \frac{\partial f_k}{\partial x_j} \delta x_j = 0$ are conditions for the components of the displacement δx_j.

They may be written in the form:

$$\frac{\partial f_1}{\partial x_1} \delta x_1 + \frac{\partial f_1}{\partial x_2} \delta x_2 + \ldots + \frac{\partial f_1}{\partial x_{3n}} \delta x_{3n} = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1.136)$$
$$\frac{\partial f_p}{\partial x_1} \delta x_1 + \frac{\partial f_p}{\partial x_2} \delta x_2 + \ldots + \frac{\partial f_p}{\partial x_{3n}} \delta x_{3n} = 0$$

The matrix of the above set of equations is the Jacobi matrix of p independent functions $f_1(x,t),\ldots,f_p(x,t)$. As we know there exists a determinant of p order different from zero in the Jacobi matrix of independent functions (p is the number of equations of constraints). Let the first p columns of the Jacobi form the determinant different from zero. Let us rewrite the equations (1.136) as follows:

$$\frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_1}{\partial x_p}\delta x_p = -\frac{\partial f_1}{\partial x_{p+1}}\delta x_{p+1} - \ldots - \frac{\partial f_1}{\partial x_{3n}}\delta x_{3n}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1.136a)$$

$$\frac{\partial f_p}{\partial x_1}\delta x_1 + \frac{\partial f_p}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_p}{\partial x_p}\delta x_p = -\frac{\partial f_p}{\partial x_{p+1}}\delta x_{p+1} - \ldots - \frac{\partial f_p}{\partial x_{3n}}\delta x_{3n}$$

The first p components of the virtual displacement δx can be calculated if the remaining 3n-p components are known, so the first p components of the displacement δx are not arbitrary any longer. The equations (1.135) can be rewritten in the form:

$$\sum_{j=1}^{p}\left(X_j + \sum_{k=1}^{p}\lambda_k \frac{\partial f_k}{\partial x_j} - m_j \ddot{x}_j\right)\delta x_j + \sum_{j=p+1}^{3n}\left(X_j + \sum_{k=1}^{p}\lambda_k \frac{\partial f_k}{\partial x_j} - m_j \ddot{x}_j\right)\delta x_j = 0 \qquad (1.137)$$

Because the first p elements of the above equation contain dependent components of the virtual displacement δx_k, we shall try to make them zero by choosing such values of the coefficients λ_k for which each of the equations below is satisfied for j=1,…,p.

$$X_j + \sum_{k=1}^{p}\lambda_k \frac{\partial f_k}{\partial x_j} - m_j \ddot{x}_j = 0 \qquad (1.138)$$

In other words we ask if the following set of equations for $\lambda_1,\ldots,\lambda_p$ can be solved:

$$\lambda_1 \frac{\partial f_1}{\partial x_1} + \lambda_2 \frac{\partial f_2}{\partial x_1} + \ldots + \lambda_p \frac{\partial f_p}{\partial x_1} = m_1 \ddot{x}_1 - X_1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (1.139)$$

$$\lambda_1 \frac{\partial f_1}{\partial x_p} + \lambda_2 \frac{\partial f_2}{\partial x_p} + \ldots + \lambda_p \frac{\partial f_p}{\partial x_p} = m_p \ddot{x}_p - X_p$$

Yes, it can be solved (!) because the matrix of coefficients

$$\begin{Bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \ldots & \frac{\partial f_p}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \ldots & \frac{\partial f_p}{\partial x_2} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial f_1}{\partial x_p} & \frac{\partial f_2}{\partial x_p} & \ldots & \frac{\partial f_p}{\partial x_p} \end{Bmatrix} \qquad (1.140)$$

is the transpose of the matrix of set (1.136a). As a result it remains from (1.135):

$$\sum_{j=p+1}^{3n}\left(X_j + \sum_{k=1}^{p}\lambda_k \frac{\partial f_k}{\partial x_j} - m_j \ddot{x}_j\right)\delta x_j = 0 \qquad (1.141)$$

The above set of equations has to be satisfied for arbitrary δx_j for j=p+1,…,3n. This can be possible only for the sum in brackets equal to zero, so we have:

$$\begin{aligned} m_j \ddot{x}_j &= X_j + \sum_{j=1}^{3n}\lambda_k \frac{\partial f_k}{\partial x_j} \\ f_k(x,t) &= 0 \end{aligned} \qquad \text{for j=1,…3n and k=1,…,p} \qquad (1.142)$$

The above system of 3n+p equations for 3n+p unknowns (3n coordinates and p λ_k coefficients) is the system of Lagrange equations of the first kind. The expression $\sum_{j=1}^{3n} \lambda_k \frac{\partial f_k}{\partial x_j}$ represents the constrain forces.

EXAMPLE: Let us find the motion of a point-like particle on side-surface of a vertical cylinder. The radius of the cylinder base changes as $\rho=\rho_0+\rho_1 t$.

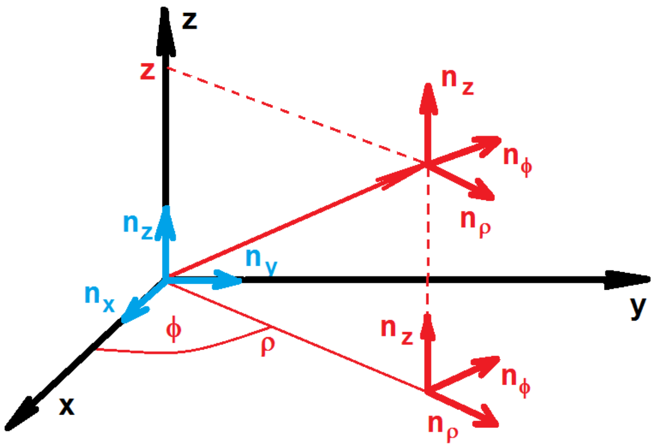

Fig.2.1. Cylindrical coordinates describe position of a point with the radius ρ, the angle ϕ and the coordinate z. The position of a point in the cylindrical coordinates is given by $\vec{r} = \rho\vec{n}_\rho + z\vec{n}_z$.

The relation between the versors of Cartesian coordinates and cylindrical coordinates are given by:

$$\vec{n}_\rho = \vec{n}_x \cos\varphi + \vec{n}_y \sin\varphi$$

$$\vec{n}_\varphi = -\vec{n}_x \sin\varphi + \vec{n}_y \cos\varphi$$

$$\vec{n}_z = \vec{n}_z$$

The versors $\vec{n}_\rho$ *and* $\vec{n}_\varphi$ *change their direction in time, the versor* $\vec{n}_z$ *is constant. The velocity of a point-like particle in the cylindrical coordinates is given by:*

$$\vec{v} = \dot{\vec{r}} = \dot{\rho}\vec{n}_\rho + \rho\dot{\vec{n}}_\rho + \dot{z}\vec{n}_z$$

Taking into account that $\dot{\vec{n}}_\rho = \dot{\varphi}\vec{n}_\varphi$ *we get:*

$$\dot{\vec{r}} = \dot{\rho}\vec{n}_\rho + \rho\dot{\varphi}\vec{n}_\varphi + \dot{z}\vec{n}_z$$

so the cylindrical components of velocity are:

$$v_\rho = \dot{\rho}$$

$$v_\varphi = \rho\dot{\varphi}$$

$$v_z = \dot{z}$$

In the same way we can calculate the cylindrical coordinates of acceleration:

$$a_\rho = \ddot{\rho} - \rho\dot{\varphi}^2$$

$$a_\varphi = \frac{1}{\rho}\frac{d}{dt}\left(\rho^2\dot{\varphi}\right)$$

$$a_z = \dot{z}$$

Lagrange equations are as follows:

$$m\left(\ddot{\rho} - \rho\dot{\varphi}^2\right) = 0 + \lambda$$

$$m\frac{1}{\rho}\frac{d}{dt}\left(\rho^2\dot{\varphi}\right) = 0$$

$$m\ddot{z} = -mg$$

$$\rho = \rho_0 + \rho_1 t$$

And finally we get:

$$\varphi = \int \frac{const}{\left(\rho_0 + \rho_1 t\right)} dt$$

$$z = -\frac{gt^2}{2} + \dot{z}_0 t + z_0$$

$$\rho = \rho_0 + \rho_1 t$$

2.2. Lagrange equations of the second kind

Let us consider the example of a plane mathematical pendulum (Fig.2.2).

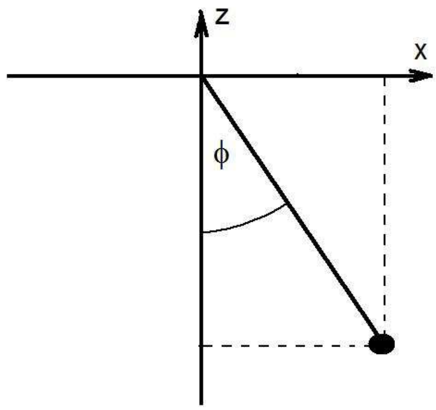

Fig.2.2. Mathematical pendulum. Its position can be described both by the Cartesian coordinates x,z and by the angle ϕ.

The motion of such a plane pendulum is restricted by equations of constraints:

$$x^2 + z^2 - l^2 = 0 \qquad (2.11)$$

$$y = 0$$

The number of degrees of freedom of such a pendulum is equal to f=3-2=1. The position of the system can be given either by one of the x,z coordinates or by the angle ϕ. The Cartesian coordinates are related to the angle ϕ by:

$$x = l\sin\phi$$
$$z = -l\cos\phi \qquad (2.12)$$

Let us note that when we substitute (2.12) to (2.11) we obtain identity :

$$l^2\sin^2\phi + l^2\cos^2\phi \equiv l^2$$

The above simple example can be **generalized to** a system of n point-like particles. Let a system of point-like particles be subject to constraints given by equations $f_k(x,t)$, k=1,…,p. The number of degrees of freedom is equal to f=3n-p. Let us suppose that there exist f parameters $q=[q_1,\dots,q_{3n-p}]$ which define the position of the system in space. If so, there have to exist equations:

$$x_j = x_j(q,t) \qquad \text{for } j=1,\dots,3n \qquad (2.13)$$

If the relations (2.13) substituted to the equations of constraints give as a result the following identity:

$$f_k(x(g,t),t) \equiv 0 \qquad (2.14)$$

the parameters $q=[q_1,\dots,q_f]$ are **the generalized coordinates** of the system of point-like particles. In consequence of the identity (2.14) we obtain:

$$\frac{\partial f_k}{\partial q_l} = 0 \qquad \text{for } k=1,\dots,p \text{ and } l=1,\dots,3n \qquad (2.15)$$

We shall prove the following two identities, important for our further considerations:

$$\frac{\partial \dot{x}_j}{\partial \dot{q}_l} = \frac{\partial x_j}{\partial q_l} \qquad (2.16)$$

$$\frac{\partial \dot{x}_j}{\partial q_l} = \frac{d}{dt}\frac{\partial x_j}{\partial q_l} \qquad (2.17)$$

Because each configuration coordinate is in general function of all generalized coordinates and time $x_j=x_j(q_1,\dots,q_{3n-p},t)$ we have:

$$\dot{x}_j = \sum_{s=1}^{f}\frac{\partial x_j}{\partial q_s}\dot{q}_s + \frac{\partial x_j}{\partial t} \qquad (2.18)$$

Differentiating (2.18) with respect to $\dot{q}_l$ for fixed l we get:

$$\frac{\partial \dot{x}_j}{\partial \dot{q}_l} = \frac{\partial x_j}{\partial q_l} \qquad (2.19)$$

so (2.16) is proved. The derivative $\frac{d}{dt}\frac{\partial x_j}{\partial q_l}$ is equal to:

$$\frac{d}{dt}\frac{\partial x_j}{\partial q_l} = \sum_{k=1}^{f}\frac{\partial^2 x_j}{\partial q_k \partial q_l}\dot{q}_k + \frac{\partial^2 x_j}{\partial t \partial q_l} \qquad (2.20)$$

Differentiating (2.18) with respect to fixed q_l we get:

$$\frac{\partial \dot{x}_j}{\partial q_l} = \sum\left(\frac{\partial^2 x_j}{\partial q_l \partial q_s}\dot{q}_s + \frac{\partial x_j}{\partial q_s}\frac{\partial \dot{q}_s}{\partial q_l}\right) + \frac{\partial^2 x_j}{\partial q_l \partial t} \qquad (2.21)$$

The quantities $\dot{q}_l$ and q_l are independent which means that $\partial\dot{q}_s/\partial q_l = 0$, so the second component is equal to zero. If so, the equations (2.20) and (2.21) become identical and relation (2.17) is proved.

2.2.1. D'Alembert Principle in generalized coordinates

The Cartesian coordinates are a function of f=3n-p generalized coordinates (and sometimes of time) $x_j = x_j(q_1, q_2, ..., q_f, t)$. Components of the virtual displacement at a fixed moment of time can be expressed as follows:

$$\delta x_j = \sum_{l=1}^{f} \frac{\partial x_j}{\partial q_l} \delta q_l \qquad (2.22)$$

From mathematical point of view the equation (2.22) is a variant (at a fixed time) of Cartesian coordinates[5]. The f-dimensional vector $\delta q = [\delta q_1, \delta q_2, \ldots, \delta q_f]$ is a generalized virtual displacement. As we know, a virtual displacement in 3n-dimensional configuration space has to satisfy the p following conditions:

$$\sum_{j=1}^{3n} \frac{\partial f_k}{\partial x_j} \delta x_j = 0 \qquad \text{for } k=1,2,\ldots,p \qquad (2.23)$$

Let us find the conditions to be satisfied by components of a generalized virtual displacement.

$$\sum_{j=1}^{3n} \frac{\partial f_k}{\partial x_j} \delta x_j = \sum_{j=1}^{3n} \frac{\partial f_k}{\partial x_j} \sum_{l=1}^{f} \frac{\partial x_j}{\partial q_l} \delta q_l = \sum_{l=1}^{f} \left(\sum_{j=1}^{3n} \frac{\partial f_k}{\partial x_j} \frac{\partial x_j}{\partial q_l} \right) \delta q_l = \qquad (2.24)$$
$$= \sum_{l=1}^{f} \frac{\partial f_k}{\partial q_l} \delta q_l = 0$$

According to equation (2.15) all derivatives $\partial f_k / \partial q_l = 0$. This means that equations (2.24) are no restricting conditions on the generalized virtual displacement. When motion of a system is described in f-dimensional generalized space, the description of such a motion becomes similar to description of motion of a free system.

Derivation of Lagrange equations

The d'Alembert principle in a configuration space is of the form:

$$\sum_{j=1}^{3n} \left(X_j - m_j \ddot{x}_j \right) \delta x_j = 0$$
$$f_k(x, t) = 0$$
$$\sum \frac{\partial f_k}{\partial x_j} \delta x_j = 0$$

Let us substitute $\delta x_j = \sum_{l=1}^{f} \frac{\partial x_j}{\partial q_l} \delta q_l$ in the first equation of d'Alembert principle. We get:

$$\sum_{j=1}^{3n} \left(X_j - m_j \ddot{x}_j \right) \delta x_j = \sum_{j=1}^{3n} \left(X_j - m_j \ddot{x}_j \right) \sum_{l=1}^{f} \frac{\partial x_j}{\partial q_l} \delta q_l = \sum_{l=1}^{f} \sum_{j=1}^{3n} \left(X_j - m_j \ddot{x}_j \right) \frac{\partial x_j}{\partial q_l} \delta q_l =$$
$$\sum_{l=1}^{f} \left(\sum_{j=1}^{3n} X_j \frac{\partial x_j}{\partial q_l} - \sum_{j=1}^{3n} m_j \ddot{x}_j \frac{\partial x_j}{\partial q_l} \right) \delta q_l = 0$$

Definition:

$$Q_l = \sum_{j=1}^{3n} X_j \frac{\partial x_j}{\partial q_l} \qquad (2.25)$$

[5] The term „variant" is used to underline that the variables $q_1, q_2, \ldots, q_f$ are also functions of time representing a trajectory of a point-like particle in f-dimensional generalized space. Reader can find the details in any textbook on calculus of variations.

We get:

$$\sum_{l=1}^{f}\left(Q_l-\sum_{j=1}^{3n}m_j\ddot{x}_j\frac{\partial x_j}{\partial q_l}\right)\delta q_l=0 \tag{2.26}$$

Let us focus on the second component of the above equation. For a start let us calculate:

$$\frac{d}{dt}\sum_{j=1}^{3n}m_j\dot{x}_j\frac{\partial x_j}{\partial q_l}=\sum_{j=1}^{3n}m_j\ddot{x}_j\frac{\partial x_j}{\partial q_l}+\sum_{j=1}^{3n}m_j\dot{x}_j\frac{d}{dt}\frac{\partial x_j}{\partial q_l} \tag{2.26}$$

so we have:

$$\sum_{j=1}^{3n}m_j\ddot{x}_j\frac{\partial x_j}{\partial q_l}=\frac{d}{dt}\sum_{j=1}^{3n}m_j\dot{x}_j\frac{\partial x_j}{\partial q_l}-\sum_{j=1}^{3n}m_j\dot{x}_j\frac{d}{dt}\frac{\partial x_j}{\partial q_l}=$$

$$=\frac{d}{dt}\sum_{j=1}^{3n}m_j\dot{x}_j\frac{\partial \dot{x}_j}{\partial \dot{q}_l}-\sum_{j=1}^{3n}m_j\dot{x}_j\frac{\partial \dot{x}_j}{\partial q_l}=\frac{d}{dt}\sum_{j=1}^{3n}m_j\frac{1}{2}\frac{\partial \dot{x}_j^2}{\partial \dot{q}_l}-\sum_{j=1}^{3n}m_j\frac{1}{2}\frac{\partial \dot{x}_j^2}{\partial q_l}=$$

$$\frac{d}{dt}\sum_{j=1}^{3n}\frac{\partial\left(\frac{m_j\dot{x}_j{}^2}{2}\right)}{\partial \dot{q}_l}-\sum_{j=1}^{3n}\frac{\partial\left(\frac{m_j\dot{x}_j{}^2}{2}\right)}{\partial q_l}=\frac{d}{dt}\frac{\partial}{\partial \dot{q}_l}\sum_{j=1}^{3n}\frac{m_j\dot{x}_j{}^2}{2}-\frac{\partial}{\partial q_l}\sum_{j=1}^{3n}\frac{m_j\dot{x}_j{}^2}{2}$$

The expression $\sum_{j=1}^{3n}\frac{m_j\dot{x}_j{}^2}{2}$ is the kinetic energy of the system. Denoting the energy with T we obtain:

$$\sum_{j=1}^{3n}m_j\ddot{x}_j\frac{\partial x_j}{\partial q_l}=\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_l}-\frac{\partial T}{\partial q_l} \tag{2.27}$$

and finally we get:

$$\sum_{l=1}^{f}\left(Q_l-\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_l}+\frac{\partial T}{\partial q_l}\right)\delta q_l=0 \tag{2.28}$$

The above equation has to be satisfied for arbitrary values components of the generalized virtual displacement δq_l. This is possible only when for all values of l=1,…,f we have:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}}-\frac{\partial T}{\partial q_l}=Q_l \qquad \text{for l-1,…,f} \tag{2.29}$$

Lagrange equations for a potential field

If motion of a system of point-like particles takes place in a potential field, components of forces are given by:

$$X_j=-\frac{\partial V(x,t)}{\partial x_j} \tag{2.30}$$

The generalized forces are as follows:

$$Q_l=\sum_{j=1}^{3n}X_j\frac{\partial x_j}{\partial q_l}=-\sum_{j=1}^{3n}\frac{\partial V}{\partial x_j}\frac{\partial x_j}{\partial q_l}=-\frac{\partial V}{\partial q_l} \tag{2.31}$$

where the potential V is expressed by the generalized coordinates V(x,t)=V(x(q,t),t)=V(q,t). The Lagrange equations can be written as:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_l}-\frac{\partial T}{\partial q_l}=-\frac{\partial V}{\partial q_l} \qquad \text{l=1,…,f} \tag{2.32}$$

Rearranging the above equation we get:

$$\frac{d}{dt}\frac{\partial(T-V)}{\partial \dot{q}_l}-\frac{\partial(T-V)}{\partial q_l}=0 \qquad \text{l=1,…,f} \tag{2.33}$$

The expression T-V is denoted as L and called the Lagrangian function (or just Lagrangian). Finally we get for a system of point-like particles moving in a potential field:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l}-\frac{\partial L}{\partial q_l}=0 \qquad (2.34)$$

The general solutions of f Lagrange equations can be written in the form:

$$q_l = q_l\left(t, C_1, ..., C_{2f}\right) \quad l=1,\ldots,f=3n\text{-}p \qquad (2.34a)$$

where $C_1,\ldots C_{2f}$ are 2f constants dependent on initial conditions.

2.3. Invariants of Lagrange equations

The expression $\partial L / \partial \dot{q}_l = p_l$ defines the l-component of generalized momentum. If so, the Lagrange equations can be written in the form:

$$\frac{dp_l}{dt}=\frac{\partial L}{\partial q_l}=\frac{\partial (T-V)}{\partial q_l}=-\frac{\partial V}{\partial q_l}=Q_l \qquad (2.35)$$

So we get:

$$\frac{dp_l}{dt}=Q_l \qquad (2.36)$$

The above equation resembles the second Newton's Law. Let us find the relation between the generalized momentum and the "ordinary" momentum in Cartesian coordinates in a potential field. The Lagrange function can be written as:

$$L = L(q,\dot{q},t) = L(x(q,t), \dot{x}(q,\dot{q},t), t) \qquad (2.37)$$

so the components of generalized momentum can be written as:

$$p_l = \frac{\partial L}{\partial \dot{q}_l} = \sum_{j=1}^{3n}\frac{\partial L}{\partial \dot{x}_j}\frac{\partial \dot{x}_j}{\partial \dot{q}_l} = \sum_{j=1}^{3n}\frac{\partial L}{\partial \dot{x}_j}\frac{\partial x_j}{\partial q_l} \qquad (2.38)$$

Taking into account that:

$$\frac{\partial L}{\partial \dot{x}_j} = \frac{\partial (T-V)}{\partial \dot{x}_j} = \frac{\partial}{\partial \dot{x}_j}\sum_{s=1}^{3n}\frac{m_s}{2}\dot{x}_s^{\,2} = m_j\dot{x}_j = p_{x_j} \qquad (2.39)$$

We get:

$$p_l = \sum_{j=1}^{3n} p_{x_j}\frac{\partial x_j}{\partial q_l} \qquad (2.40)$$

As we see the relation between the components of generalized momentum and the components of ordinary momentum is similar to the relation between the generalized force and "ordinary" force defined by equation (2.25).

2.3.1. Cyclic coordinates

Let us remind the Lagrange equations in the form:

$$\frac{dp_l}{dt}=\frac{\partial L}{\partial q_l} \qquad (2.41)$$

Let us assume that the Lagrange function $L(q,\dot{q},t)$[6] does not depend on one of the generalized coordinates q_s. Such a generalized coordinate is called a cyclic coordinate. In consequence:

$$\frac{dp_s}{dt}=\frac{\partial L}{\partial q_s}=0 \qquad \Rightarrow \qquad p_s = const \qquad (2.42)$$

[6] Let us remember that $q=\lfloor q_1,\ldots,q_f \rfloor$ represents all f=3n-p generalized coordinates, $\dot{q}=\lfloor \dot{q}_1,\ldots,\dot{q}_f \rfloor$ represents all f generalized components of speed.

As we see the component of generalized momentum corresponding to a cyclic coordinate is constant in time.

Symmetry in Physics

We say that a system is symmetrical when there exists such a transformation of the system which transforms it to itself. For instance if a system transfers to itself after reflection across a plane we say that the system posses a mirror symmetry. If a physical structure transforms into itself when rotated round an axis we say it possess an axis of symmetry. Transformation into itself leads to conclusion that the physical properties of a system are independent of the transformation. Because the mechanical properties of a physical system are defined by its Lagrangian it is obvious that Lagrangian must be independent of any transformation which transforms a system into itself. In other words any symmetry should be reflected by independence of Lagrange function of some parameter describing the symmetry. There are three main symmetries of physical space and time which are of crucial importance for analytical mechanics, namely translational symmetry of space (Lagrange function is independent of point of space), rotational symmetry (Lagrange function is independent of rotation around some axis) and translational symmetry in time (Lagrange function is not explicit function of time) .

2.3.1.1. Lagrange function is independent of position in space (space is homogeneous)

For a homogeneous space the Lagrange function does not depend on a radius vector, i.e. the Lagrange function does not depend on Cartesian coordinates x,y, and z. In other words we have to do with translational symmetry of homogeneous physical space. If so, for a point-like particle the Lagrange function is of the form:

$$L = T - V_0 = L(\dot{x}, \dot{y}, \dot{z}, t) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \qquad (2.43)$$

The coordinates x,y and z are cyclic coordinates, so the components of momentum corresponding to the coordinates remain constant. So we have:

$$\begin{aligned} \frac{\partial L}{\partial x} &= 0 \Rightarrow \frac{dp_x}{dt} = 0 \Rightarrow p_x = const \\ \frac{\partial L}{\partial y} &= 0 \Rightarrow \frac{dp_y}{dt} = 0 \Rightarrow p_y = const \\ \frac{\partial L}{\partial z} &= 0 \Rightarrow \frac{dp_z}{dt} = 0 \Rightarrow p_z = const \end{aligned} \qquad (2.44)$$

CONCLUSION: Law of conservation of momentum results from translational symmetry of homogeneous physical space.

2.3.1.2. Lagrange function is independent of rotation in space (space is isotropic)

Let us calculate the Lagrange function of a free point-like particle using spherical coordinates (see Fig.2.3).

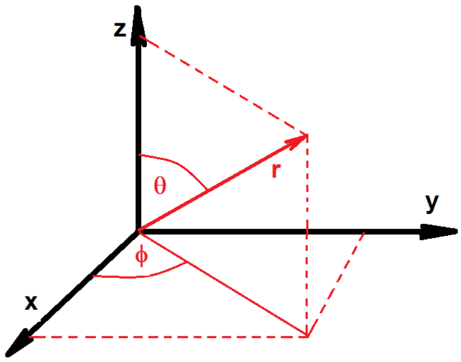

Fig.2.3. Position of a point is given by length of radius vector r and two angles ϕ and θ.

The relation between Cartesian coordinates and spherical coordinates are given by:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi \qquad (2.45)$$
$$z = r\cos\theta$$

The kinetic energy of a point-like particle is therefore given by:

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) \qquad (2.46)$$

Let us note that the kinetic energy depends on two of three spherical coordinates r and θ and is independent of the third coordinate ϕ. Let us assume that the potential energy is also independent of the angle ϕ. We therefore have:

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - V(r,\theta) \qquad (2.47)$$

The Lagrange function is independent of rotations around the z-axis. We say that the physical space is isotropic with respect to these rotations. The angle ϕ is a cyclic coordinate, so the generalized momentum corresponding to the angle ϕ is constant.

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = mr^2\sin^2\theta\dot{\phi} = mr_{xy}r_{xy}\dot{\phi} = mv_\phi r_{xy} = J_z \qquad (2.47)$$

As we see conservation of angular momentum results from isotropy of physical space.

2.3.1.3. Lagrange function is not explicit function of time

Let us multiply Lagrange equations

$$\dot{p}_l = \frac{\partial L}{\partial q_l} \qquad /\cdot\dot{q}_l\sum_{l=1}^{3n} \qquad (2.48)$$

by $\dot{q}_l$ and summarize over l. We obtain:

$$\sum_{l=1}^{f}\left(\dot{p}_l\dot{q}_l - \frac{\partial L}{\partial q_l}\dot{q}_l\right) = 0 \qquad (2.49)$$

Let us calculate the derivative:

$$\frac{d}{dt}(p_l\dot{q}_l) = \dot{p}_l\dot{q}_l + p_l\ddot{q}_l \qquad (2.50)$$

Using (2.50) we get:

$$\sum_{l=1}^{f}\left(\frac{d}{dt}(p_l\dot{q}_l) - p_l\ddot{q}_l - \frac{\partial L}{\partial q_l}\dot{q}_l\right) = \sum_{l=1}^{f}\left(\frac{d}{dt}(p_l\dot{q}_l) - \frac{\partial L}{\partial\dot{q}_l}\ddot{q}_l - \frac{\partial L}{\partial q_l}\dot{q}_l\right) \qquad (2.51)$$

The total time derivative of Lagrangian is as follows:

$$\frac{dL}{dt} = \sum_l \left(\frac{\partial L}{\partial q_l} \dot{q}_l + \frac{\partial L}{\partial \dot{q}_l} \ddot{q}_l \right) + \frac{\partial L}{\partial t} \quad (2.52)$$

Taking into account (2.52) we get from (2.51):

$$\sum_{l=1}^{f} \frac{d}{dt}(p_l \dot{q}_l) - \frac{dL}{dt} - \frac{\partial L}{\partial t} = 0 \quad (2.53)$$

Rearranging (2.53) we obtain:

$$\frac{d}{dt}\left(\sum_{l=1}^{f} p_l \dot{q}_l - L \right) = -\frac{\partial L}{\partial t} \quad (2.54)$$

Let us denote $G = \sum_{l=1}^{f} p_l \dot{q}_l - L$. We get:

$$\frac{dG}{dt} = -\frac{\partial L}{\partial t} \quad (2.55)$$

It results that if Lagrange function is not explicit function of time, i.e. if $\partial L / \partial t = 0$ the quantity G remains constant in time. We shall show that under some assumptions G is total energy of a system.

THEOREM: If the configuration coordinates as functions of generalized coordinates are not explicit functions of time, i.e. if $x_j = x_j(q)$ and we have to do with motion in a potential field, i.e. V=V(q,t), then function G is the total energy of a system, i.e. **G=T+V.**

In order to prove the above theorem, we shall accept two additional theorems.

AUXILIARY THEOREM 1**: If** $x_j = x_j(q)$ **then** $T = \sum_{l=1}^{f} \sum_{k=1}^{f} a_{lk} \dot{q}_l \dot{q}_k$ **, i.e. the kinetic energy is a bilinear form of generalized speeds.**

The kinetic energy of a system of n point-like particles is given by $T = \sum_{j=1}^{3n} \frac{m_j}{2} \dot{x}_j^{\,2}$. Taking into account that $\dot{x}_j = \sum_{l=1}^{f} \frac{\partial x}{\partial q_l} \dot{q}_l$ we get:

$$T = \sum_{j=1}^{3n} \frac{m_j}{2} \sum_{l=1}^{f} \frac{\partial x_j}{\partial q_l} \dot{q}_l \sum_{k=1}^{f} \frac{\partial x_j}{\partial q_k} \dot{q}_k = \sum_{j=1}^{3n} \sum_{l=1}^{f} \sum_{k=1}^{f} \frac{m_j}{2} \frac{\partial x_j}{\partial q_l} \frac{\partial x_j}{\partial q_k} \dot{q}_l \dot{q}_k \quad (2.56)$$

Because the result of summation is independent of sequence of summation, we obtain:

$$T = \sum_{l=1}^{f} \sum_{k=1}^{f} \left(\sum_{j=1}^{3n} \frac{m_j}{2} \frac{\partial x_j}{\partial q_l} \frac{\partial x_j}{\partial q_k} \right) \dot{q}_l \dot{q}_k = \sum_{l=1}^{f} \sum_{k=1}^{f} a_{lk} \dot{q}_l \dot{q}_k \quad (2.57)$$

The sum $\left(\sum_{j=1}^{3n} \frac{m_j}{2} \frac{\partial x_j}{\partial q_l} \frac{\partial x_j}{\partial q_k} \right)$ depends on the indexes l and k, so it can be denoted as a_{lk}.

AUXILIARY THEOREM 2**: If $f(x_1,\ldots,x_n)$ is a homogeneous polynomial of degree m then**

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i = mf \quad (2.58)$$

The G function can be written as:

$$G = \sum_{l=1}^{f} p_l \dot{q}_l - L = \sum_{l=1}^{f} \frac{\partial L}{\partial \dot{q}_l} \dot{q}_l - L \quad (2.59)$$

L=T-V(q,t), so the derivative $\frac{\partial L}{\partial \dot{q}_l} = \frac{\partial T}{\partial \dot{q}_l}$, so we get:

$$G = \sum_{l=1}^{f} \frac{\partial T}{\partial \dot{q}_l} \dot{q}_l - L = 2T - T + V = T + V \qquad (2.60)$$

because T is a homogeneous function of degree 2, so $\sum_{l=1}^{f} \frac{\partial T}{\partial \dot{q}_l} \dot{q}_l = 2T$.

2.4. The action principle

We derived the Lagrange's equations from Newton's equations of motion. This is not the only way to get the Lagrange's equations. The equations can be obtained in another, very general way. In order to show this alternative way to get the Lagrange's equations we must get familiar with some concepts of calculus of variations.

2.4.1. Functions and functionals

A function is a relation between a set of inputs and a set of permissible outputs assuming that each input is related to exactly one output. The input to a function is called an

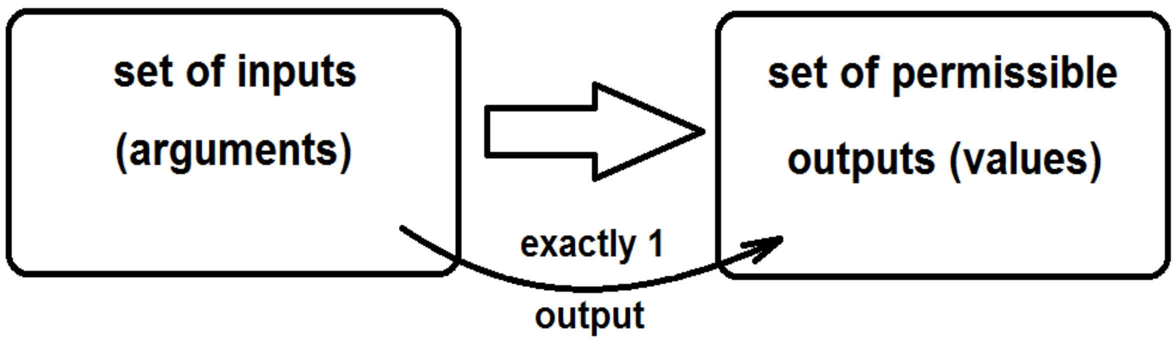

Fig.2.4. FUNCTION: relation between a set of inputs and a set of permissible outputs. Each input related to one output.

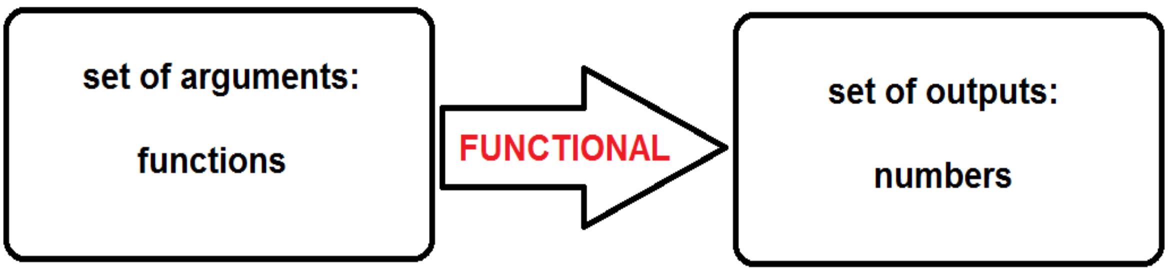

Fig.2.5. In case of functional arguments are functions.

argument and the output is called the value of function. In case of functions both the set of arguments and the set of values are numbers.

When the set of arguments is a set of functions, and a set of values is a set of numbers we have to do with a functional. Definite integral

$$F[f(x)] = \int_a^b f(x)dx \qquad (2.61)$$

is a good example of a functional. We often use square brackets in order to emphasis the fact that functional F is a function of functions.

2.4.2. The action principle

Let us suppose we have a system of point-like particles subject to holonomic constraints. Let the number of degrees of freedom be f=3n-p, so a motion of the system is given by f generalized coordinates $q_l(t)$ (f=1,…,f). The Lagrange function of such a system is given by $L(q(t),\dot{q}(t),t)$. The action for such a motion is defined:

$$I[q(t)] = \int_{t_0}^{t} L(q(t),\dot{q}(t),t)dt \qquad (2.62)$$

The action is a functional, the set of arguments are functions $q_1(t),\dots q_f(t)$. Among many functions describing possible motion between two fixed point of time t_0 and t, one set of functions q_f refers to the path actually taken between the two points. For this set of functions the functional I[q(t)] has its minimum.

$$I[q(t)] = \int_{t_0}^{t} L(q(t),\dot{q}(t),t)dt...has.its..\min..\Rightarrow ..q(t).correspond.to.real.motion$$

It results from detailed considerations of calculus of variations that the functional (2.62) has its minimum for Lagrange functions satisfying the Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_l} - \frac{\partial L}{\partial q_l} = 0$$

3. PHASE SPACE, HAMILTON'S EQUATIONS

Position of a system of n point-like particles in 3D space can be defined by their 3n configuration coordinates x_j or by f=3n-p generalized coordinates q_l. However, in order to define STATE of a system to be able to foresee its development in time, we have to add velocities of all particles at a certain point of time. Instead of velocities momenta of particles can be also taken. The set of coordinates (or generalized coordinates in case of restricted system) and momenta of all points form so-called phase space. Phase space is very important concept of both mechanics and statistical physics. Motion of a system in phase space can be found by solving Hamilton's equations.

Phase coordinates (q_l,p_l) and Hamilton's equations lead to the Hamiltonian formulation of mechanics. Hamilton's equations, when compared to Lagrange's equations, do not present much, if any, advantage when it is used to solve problems of Newtonian mechanics. However, Hamilton's formulation of mechanics is necessary to build and study quantum mechanics.

3.1. Hamilton's equations

The function $G=\sum_{l=1}^{f} p_l \dot{q}_l - L$ defined previously is a function of coordinates, velocities, momenta and sometimes it can be explicit function of time. Let us write the function in the form:

$$G=\sum_{l=1}^{f} p_l v_l - L \tag{3.1}$$

MATHEMATICAL INSERT: Let us take 3n functions $x_j(q_1,\ldots,q_f,t)$ describing the relations between 3n configuration coordinates and 3n-p generalized coordinates. In general the functions can be explicit functions of time, though we are interested in the case $x_j(q_1,\ldots,q_f)$ first of all. The differential (variation) of the functions is:

$$\delta x_j=\sum_{l=1}^{f}\frac{\partial x_j}{\partial q_l}\delta q_l+\frac{\partial x_j}{\partial t}dt \tag{3.2}$$

Symbols δ instead of d are used to emphasize that variables q_l are functions of time. The expression (3.2) is called the variation of coordinates x_j with variation of time. When we want to have a variation of coordinates x_j for a fixed point of time we get:

$$\delta x_j=\sum_{l=1}^{f}\frac{\partial x_j}{\partial q_l}\delta q_l \tag{3.3}$$

The above expression is called a variation of x_j without variation of time[7].

Let us calculate the variation of function $G=\sum_{l=1}^{f} p_l v_l - L$. We get:

$$\delta G=\delta\sum_{l=1}^{f} p_l v_l-\delta L=\sum_{l=1}^{f}\delta(p_l v_l)-\sum_{l=1}^{f}\left(\frac{\partial L}{\partial q_l}\delta q_l+\frac{\partial L}{\partial v_l}\delta v_l\right)=$$

$$=\sum_{l=1}^{f}(p_l\delta v_l+v_l\delta p_l)-\sum_{l=1}^{f}(\dot{p}_l\delta q_l+p_l\delta v_l)=\sum_{l=1}^{f}(v_l\delta p_l-\dot{p}_l\delta q_l)=\sum(\dot{q}_l\delta p_l-\dot{p}_l\delta q_l) \tag{3.4}$$

[7] Variational calculus is an important part of mathematics, useful for so called variational principles of mechanics. The calculus of variations deals with the study of extremum values of functions (called functionals) depending on another function. Reader can find many textbooks on variational calculus. For the purposes of this lecture we will use some simple analogies between functions and functionals.

So far we used the function G as a function of generalized coordinates q_l, generalized velocities $\dot{q}_l = v_l$, generalized momenta $p_l = \partial L / \partial \dot{q}_l$ and sometimes time t. Let us note that it should be possible to replace the components of generalized velocities v_l by components of generalized momenta p_l using the definitions of generalized momentum $p_l = \partial L / \partial \dot{q}_l$ [8].

Replacing the velocities by momenta we get:

$$G(q, v, p, t) \Rightarrow H(q, p, t) \tag{3.5}$$

The function H(q,p,t) is called Hamilton's function. Let us calculate the variation of Hamilton's function without variation of time:

$$\delta H = \sum \left(\frac{\partial H}{\partial q_l} \delta q_l + \frac{\partial H}{\partial p_l} \delta p_l \right) \tag{3.6}$$

Let us compare the variations (3.4) and (3.6). They are variations of G and H, i.e. they are variations of the same function though written using different sets of variables. Subtracting (3.4) from (3.6) we get:

$$\delta H - \delta L = 0 = \sum_{l=1}^{f} \left(\frac{\partial H}{\partial q_l} + \dot{p}_l \right) \delta q_l + \sum_{l=1}^{f} \left(\frac{\partial H}{\partial p_l} - \dot{q}_l \right) \delta p_l \tag{3.7}$$

The equation (3.7) have to be satisfied for arbitrary variations of δq_l *and* δp_l and this is possible only when:

$$\dot{q}_l = \frac{\partial H}{\partial p_l}$$
$$- \dot{p}_l = \frac{\partial H}{\partial q_l} \tag{3.8}$$

The above equations are Hamilton's equations.

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[8] The replacement is possible in practice if we can solve the set of f=3n-p equations $p_l = \partial L / \partial \dot{q}_l$ for f variables v_l. However, even if we cannot solve the set of equations we shall assume that there exist singe-valued functions $p_l = p_l(v_1, ..., v_f)$.

4. INTEGRALS OF MOTION, POISSON'S BRACKETS

Consider a system on n point-like particles subject to holonomic constraints given by p equations with f=3n-p degrees of freedom. Let us assume that there exists a relation of the type:

$$F\left(q_1,...,q_f,\dot{q}_1,...,\dot{q}_f,t\right)=C(const.) \tag{4.1}$$

which is satisfied by any solution of Lagrange equations. Then (4.1) is called ***a first integral*** of motion or ***a constant of motion***. Any relation of the type:

$$F\left(q_1,...,q_f,t\right)=C(const.) \tag{4.2}$$

is called a second integral of motion. Second integrals are not as important for mechanical considerations as first integrals of motion.

Let us suppose that we have s distinct first integrals of motion $F_1,\ldots,F_s$. Then any function $G(F_1,\ldots,F_s)$ = const is also a first integral of motion, but the function G is not independent of functions $F_1,\ldots,F_s$.

The integration of Lagrange equations is considerably facilitated by the application of first integrals, because:

(i) first integrals offer information on physical nature of a system

(ii) in some cases first integrals express the conservation of fundamental physical quantities such as linear and angular momenta, energy.

Poisson's brackets are a way to find additional first integrals of motion if two of them is known.

4.1. Poisson's brackets

Suppose we have two functions of generalized coordinates and momenta defined in the phase space F(q,p,t) and G(q,p,t). Poisson's bracket of functions F and G is defined as:

$$(F,G)=\sum_{l=1}^{f}\left(\frac{\partial F}{\partial q_l}\frac{\partial G}{\partial p_l}-\frac{\partial F}{\partial p_l}\frac{\partial G}{\partial q_l}\right) \tag{4.3}$$

Properties of Poisson's brackets:

$$(F,G)=-(G,F) \tag{4.4}$$

$$(F,F)=0 \tag{4.5}$$

$$(F_1+F_2,G)=(F_1,G)+(F_2,G) \tag{4.6}$$

$$(F_1F_2,G)=F_1(F_2,G)+F_2(F_1,G) \tag{4.7}$$

$$\frac{\partial}{\partial t}(F,G)=\left(\frac{\partial F}{\partial t},G\right)+\left(F,\frac{\partial G}{\partial t}\right) \tag{4.8}$$

$$(F_1,(F_2,F_3))+(F_2,(F_3,F_1))+(F_3,(F_1,F_2))=0 \tag{4.9}$$

PROOF (4.7): simple

PROOF (4.8): simple

4.1.1. Equation of motion for physical quantity F(q,p,t)

Let us calculate the time derivative of function F)q,p,t):

$$\frac{dF}{dt} = \sum \left(\frac{\partial F}{\partial q_l} \dot{q}_l + \frac{\partial F}{\partial p_l} \dot{p}_l \right) + \frac{\partial F}{\partial t} \tag{4.10}$$

Using Hamilton's equations we get:

$$\frac{dF}{dt} = \sum_{l=1}^{f} \left(\frac{\partial F}{\partial q_l} \frac{\partial H}{\partial p_l} - \frac{\partial F}{\partial p_l} \frac{\partial H}{\partial q_l} \right) + \frac{\partial F}{\partial t} \tag{4.11}$$

so finally we get:

$$\frac{dF}{dt} = (F, H) + \frac{\partial F}{\partial t} \tag{4.12}$$

For F=H we obtain:

$$\frac{dH}{dt} = (H, H) + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \tag{4.13}$$

It results from (4.13) that the Hamilton function can only be explicit function of time.

4.1.2. Poisson-Jacobi theorem

Let us assume that we have two first integrals of motion F_1(q,p,t) and F_2(q,p,t). According to Poisson-Jacobi theorem Poisson's bracket of the two functions $(F_{1,}F_2)$=const.

Let us start from equation (4.9) taking F_3=H (Hamilton's function):

$$(F_1,(F_2,H)) + (F_2,(H,F_1)) + (H,(F_1,F_2)) = 0 \tag{4.14}$$

Let us calculate the total time derivative of functions F_1, F_2 using (4.12). We get:

$$\frac{dF_2}{dt} = (F_2, H) + \frac{\partial F_2}{\partial t} \quad \Longrightarrow \quad (F_2, H) = \frac{dF_2}{dt} - \frac{\partial F_2}{\partial t} = -\frac{\partial F_2}{\partial t} \tag{4.15}$$

$$\frac{dF_1}{dt} = (F_1, H) + \frac{\partial F_1}{\partial t} \quad \Longrightarrow \quad (H, F_1) = \frac{\partial F_2}{\partial t} - \frac{dF_1}{dt} = \frac{\partial F_1}{\partial t} \tag{4.16}$$

$$\frac{d(F_1,F_2)}{dt} = ((F_1,F_2),H) + \frac{\partial (F_1,F_2)}{\partial t} \tag{4.17}$$

Rearranging (4.17) we get:

$$(H,(F_1,F_2)) = \frac{\partial (F_1,F_2)}{\partial t} - \frac{d(F_1,F_2)}{dt} \tag{4.18}$$

Substituting (4.15), (4.16) and (4.18) to (4.14) we obtain:

$$\left(F_1, -\frac{\partial F_2}{\partial t} \right) + \left(F_2, \frac{\partial F_1}{\partial t} \right) + \left(\frac{\partial F_1}{\partial t}, F_2 \right) + \left(F_1, \frac{\partial F_2}{\partial t} \right) = \frac{d(F_1,F_2)}{dt} \tag{4.19}$$

and finally:

$$\frac{d(F_1,F_2)}{dt} = 0 \tag{4.20}$$